THE INTERNATIONAL CONFERENCE
ALGEBRA, LOGIC
AND THEIR APPLICATIONS

Dedicated to the 75th anniversary of Prof. Yuri Movsisyan and the 50th anniversary at Yerevan State University

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ABSTRACTS

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ON K-ISOTOPIC AND M-ISOTOPIC SEMIRINGS

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Definition 1. *A semiring is a set R equipped with two binary operations + and -, called addition and multiplication, such as:*

1. $(R, +)$ is a commutative monoid with identity element 0:

$$(a+b)+c = a+(b+c)$$

$$0+a = a+0 = a$$

$$a+n = b+a$$

2. $(R, \cdot)$ is a monoid with identity element 1:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$1 \cdot a = a \cdot 1 = a$$

3. Both multiplying left and right distribute over addition:

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c)$$

$$(a+b) \cdot c = (a \cdot c) + (b \cdot c)$$

4. Multiplication by 0 annihilates R:

$$0 \cdot a = a \cdot 0 = 0.$$

Definition 2. *A semiring $(R, +,)$ is called commutative, if $(R, \cdot)$ is commutative groupoid.*

Definition 3. *An idempotent semiring is a semiring , with identity $a + a = a$.*

Examples.

The motivating example of a semiring is aset of natural numbers N (including zero) under ordinary addition and multiplication. All these semirings are commutative.

1. The square $n \times n$ matrixes with non-negative entries form a (non-commutative) semiring under ordinary addition and multiplication of matrices. More generally, the same applies to the square matrices whithelements of any other given semiring S, and the semiring is generally non-commutative nevetheless S may be commutative.

2. If A is a commutative monoid, then the set *End (A)* of endomorphisms $f : A \to A$ form is a semiring, where addition is pointwise addition and multiplication is functional composition.

$$(f + g)(x) = f(x) + g(x),$$

$$(f \cdot g)(x) = g(f(x).$$

Zero morphism and identity are respective neutral elements.

3. If $Q(+,\cdot)$ is a semiring, then the set *End (Q)* of endomorphisms is a semiringunder with of the following operationsare:

$$(f + g)(x) = f(x) + g(x),$$

$$(f \cdot g)(x) = g(f(x).$$

4. The ideals of a ring is a semiring under addition and multiplication of ideals.

5. Any bounded, distributive lattice is a commutative, idempotent semiring under joining and meeting.

If $R = (R, +, 0)$ is a semiring, then we denote $R^+ = R(+)$.

Two groupoids on G are called isotopic if there are permutations of $G\rho\sigma$ and τ, such as for any $a, b \in G$,

$$a \circ b = (a\rho, b\sigma)\tau,$$

where, and $\circ$ denotes the operation in these two groupoids. The isotopy relation is an equivalence relation for the binary operations. An isomorphism of two binary operations defined on the same set is a special case of an isotopy (with $\rho = \sigma = \tau^{-1}$).

In about quasigroups the following results are known:

Theorem 1 (Albert,1943)**.** *Every groupoid that is isotopic to a quasigroup is a quasigroup itself.*

Theorem 2 (Albert,1943)**.** *Every quasigroup is isotopic to some loop.*

Theorem 3 (Albert,1943)**.** *If a loop (in particular, a group) is isotopic to some group, then they are isomorphic.*

Theorem 4 (Bruck)**.** *If a groupoid with identity element is isotopic to a semigroup, then they are isomorphic, that is, they are both semigroups with identity.*

In the book [1] the isotopy of rings with the same additive groups is defined by Albert in the following manner: $(Q.\Omega)$ and (Q',Ω')

$(Q,+,\cdot)$ and $(Q,+,\circ)$ rings are called K-isotopic if there exist bijective mapping $\alpha,\beta,\gamma : Q \to Q$ such as:

$$1)\quad \alpha(x,y) = \beta(x)\circ\gamma(y),$$
$$2)\quad \alpha,\beta,\gamma \in Aut[Q(+)].$$

Theorem 5 (Albert, Kurosh.)**.** *If a ring with identity elementisk-isotopic to an associative ring, then they are isomorphic.*

We introduce the following general concept of isotopy. $(Q,+_1,_1)$ and $(Q',+_2,_2)$ semirings are called K-isotopic if there exist bijective mapping $\alpha,\beta,\gamma : Q \to Q$ such as:

$$1)\quad \alpha(x\cdot_1,y) = \beta(x)\cdot_2,\gamma(y),$$
$$2)\quad \alpha,\beta,\gamma : Q(+_1) \to Q'(+_2)\quad \text{isomorphic mappings.}$$

Theorem 6. *If a ring with identity element is K-isotopic to an associative ring, then they are K-isomorphic.*

Theorem 7. *K-isotopic semirings are isomorphic. In thebook [2] the isotopy of algebras is defined as follows: (Q,Ω) and (Q',Ω') and algebras with binary operations are called M-isotopic, if there exist $\alpha,\beta,\gamma : Q \to Q', \psi : \Omega \to \Omega'$ bijective mappings, such as*

$$\alpha A(x,y) = (\psi A)(\beta x,\beta y)$$

for all $A \in \Omega$ and ψ is save operations arity.

Theorem 8. *M-isotopic semirings are isomorphic.*

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MINIMAL REGULAR DESSINS AND THEIR DEFORMATIONS

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A *dessin d'enfant* is a bicolored (connected) graph Γ embedded into a compact oriented surface $\mathcal{X}$ in such a way that the complement $\mathcal{X} \setminus \Gamma$ is homeomorphic to a disjoint union of open discs. Let X be a non-singular complex algebraic curve. A function β on X is called the *Belyi function* if β has no critical values other than 0, 1 and ∞. Then the preimage $\beta^{-1}[0,1]$ is a dessin d'enfant on X.

For a dessin with n edges the degrees of the faces, the black and white vertices constitute a triple of partitions of n which is called *passport*. In terms of the Belyi functions the passport is a set of ramification multiplicities over 0, 1 and ∞. We denote the set of dessins with passport π by $\mathcal{D}(\pi)$.

There is an action of $\mathbf{S}_3$ on Belyi functions, which transforms β to $1-\beta$, $1/\beta$, $\beta/(\beta-1)$, $1/(1-\beta)$ or $(\beta-1)/\beta$. For a dessin D the *chameleon group* is the stabilizer of the corresponding Belyi function under this action.

We consider Belyi functions of a smallest possible degree of genus g. The corresponding dessins have only one face, one black and one white vertex, so their passport is $(n|n|n)$ where $n = 2g+1$. Despite such a simple characterization, the number of these objects grows factorially:

$$\sum_{D \in \mathcal{D}(n|n|n)} \frac{1}{\#\mathrm{Aut}(D)} = \frac{(2g)!}{(g+1)(2g+1)}.$$

A relatively small subset of the *regular* minimal dessins with cyclic automorphism group $\mathbf{C}_n$ is described by the following theorem.

Theorem. (a) *For any g and $n = 2g+1$ the number of regular minimal dessins with n edges is equal to $n \prod_{p|n} (p-2)/p$.*

(b) *For $g = 0,1$ there is only one minimal dessin, it is regular and it's chameleon group is $\mathbf{S}_3$.*

(c) *For any $g \geq 2$ there are three regular minimal dessins with chameleon group $\mathbf{C}_2$. Their Belyi pairs can be written as*

$$X : y^2 = 1 - x^n, \quad \beta_1 = (y+1)/2, \quad \beta_2 = 2/(y+1), \quad \beta_3 = (y-1)/(y+1).$$

(d) *For any $g \geq 2$ the regular minimal dessins with chameleon group $\mathbf{C}_3$ exist if and only if*

$$n = 3^{\alpha_0} p_1^{\alpha_1} \dots p_m^{\alpha_m},$$

where $\alpha_0 = 0, 1$ and p_i – distinct primes, $p_i \equiv 1 \pmod 3$. The number of such dessins is 2^m and their Belyi pairs can be written as

$$X_{a,b} : x^a y^{a+b} + y^a z^{a+b} + z^a x^{a+b} = 0,$$

$$\beta_{a,b} = -\frac{y^{a+b}}{z^a x^b} = 1 + \frac{y^a z^b}{x^{a+b}}.$$

where $n = a^2 + ab + b^2$.

The statement (a) of the theorem is a particular case of a more general result on regular dessins with cyclic automorphism groups, see [1]. The curves $X_{a,b}$ also appeared in [4] and can be considered as a generalization of the famous Klein's quartic

$$x^3 y + y^3 z + z^3 x = 0.$$

Also all regular minimal Belyi pairs of prime degree $n = p$ can be written as pairs on *Lefchetz surfaces*, see [3]

$$X : y^p = x^m (x-1), \qquad \beta = x, \qquad 1 \leq m \leq p-2.$$

In this talk we will also present some non-regular minimal Belyi pairs and discuss deformations of the minimal dessins into Fried families (*Fried function* is a function with only 4 critical values).

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SECURE MULTI-PARTY COMPUTATION FOR ANONYMIZED DATA

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With the increase in digital communication and the need for organizations to delegate certain computational tasks to third-party providers, data security and privacy have become significant concerns. To preserve security of personally identifiable information during transit, data anonymization, such as: replacement, removal, masking and other techniques are used to add random noise either by involving cryptography or collision-free hashing. Anonymization minimizes the amount of publicly available and readable data both for humans and for computers meanwhile keeping the information functional for statistical data analysis and other business purposes.

Secure multi-party computation (MPC) helps when there is a need for conducting joint analysis or usage of sensitive data residing in the same or even in different stores without disclosing any participant's private inputs [1, 2]. A computation method which is selected and agreed among participants should ensure verifiability without reconstructing the secret for not to become a target for attackers. In order to demonstrate such behaviour, responsibilities on the whole secret, and not the portions of the secret itself, should be granted to parties involved in MPC. With MPC proper implementation, the attacker has to compromise all the shares in order to break into the system.

The present research is devoted to promote MPC algorithms based on non-commutative and non-associative data structures from higher algebra aimed at further hardening the reverse engineering in the upcoming era of powerful quantum attacks.

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PROBABILITY IDENTITIES IN n-TORSION GROUPS

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A 3-generated infinite group is constructed in which the probability of some fixed identity tends to 1. At the same time, this identity does not hold on the entire group. The question of the existence of such a group has been recently raised by several authors. More precisely, the n-periodic product of a free periodic group of rank 2 and an infinite cyclic group is considered. It is proved that in this product the probability of the identity $x^n = 1$ tends to 1, but it does not hold on the entire product.

ON CONSTRUCTION OF MONOGENIC FUNCTIONS WITH VALUES IN REDUCED QUATERNIONS

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Quaternionic analysis involves the analysis of quaternionic functions that are defined in open subsets of $\mathbb{R}^n$ $(n = 3,4)$ and that are solutions of generalized Cauchy-Riemann or Riesz systems. They are often called monogenic functions. Let

$$\mathbb{H} := \left\{ \mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \quad x_\ell \in \mathbb{R}, \quad \ell = 0,1,2,3 \right\}$$

be the real quaternion algebra (or Hamiltonian skew field), where the imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are subject to the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \qquad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

Evidently the real vector space $\mathbb{R}^4$ may be embedded in $\mathbb{H}$ by identifying the element $x := (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ with $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$. Consider the subset $\mathcal{A} := \mathrm{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H}$, then the real vector space $\mathbb{R}^3$ may be embedded in $\mathcal{A}$ via the identification of $x := (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathcal{A}$. It should be noted, however, that $\mathcal{A}$ is a real vectorial subspace, but not a subalgebra of $\mathbb{H}$.

Like in the complex case, $\mathbf{Sc}(\mathbf{x}) = x_0$ and $\mathbf{Vec}(\mathbf{x}) = x_1\mathbf{i} + x_2\mathbf{j}$ define the scalar and vector parts of $\mathbf{x}$. The conjugate of $\mathbf{x}$ is the reduced quaternion $\overline{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j}$, and the norm $|\mathbf{x}|$ of $\mathbf{x}$ is defined by $|\mathbf{x}| = \sqrt{\mathbf{x}\overline{\mathbf{x}}} = \sqrt{\overline{\mathbf{x}}\mathbf{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2}$, and it coincides with its Euclidean norm as a vector in $\mathbb{R}^3$. Let $B \subset \mathbb{R}^3$ denote the three-dimensional unit ball centered at the origin. We say that

$$\mathbf{f} : B \longrightarrow \mathcal{A}, \qquad \mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j}$$

is a reduced quaternion-valued function or, in other words, an $\mathcal{A}$-valued function, where $[\mathbf{f}]_\ell$ $(\ell = 0,1,2)$ are real-valued functions defined in B. For a real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ that has continuous first partial derivatives, the (reduced) quaternionic operators

$$D\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_0} + \mathbf{i}\frac{\partial \mathbf{f}}{\partial x_1} + \mathbf{j}\frac{\partial \mathbf{f}}{\partial x_2} \qquad \text{and} \qquad \overline{D}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x_0} - \mathbf{i}\frac{\partial \mathbf{f}}{\partial x_1} - \mathbf{j}\frac{\partial \mathbf{f}}{\partial x_2}$$

are called generalized Cauchy-Riemann (resp. conjugate generalized Cauchy-Riemann) operators on $\mathbb{R}^3$.

A continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is said to be monogenic if $D\mathbf{f} = 0$, which is equivalent to the system $\begin{cases} \operatorname{div} \overline{\mathbf{f}} = 0 \\ \operatorname{curl} \overline{\mathbf{f}} = 0. \end{cases}$

Let U be a harmonic function defined in an open subset Ω of $\mathbb{R}^3$. A vector-valued harmonic function V in Ω is called conjugate harmonic to U if $\mathbf{f} := U + V$ is monogenic in Ω. The pair $(U; V)$ is called a pair of conjugate harmonic functions in Ω.

We suggest an algorithm to the explicit construction of a “unique” pair of conjugate harmonic functions in $\mathbb{R}^3$.

Theorem 1. *Let U be a real-valued harmonic function defined in B. Define*

$$V_1(x) := -x_0 \int_0^1 \frac{\partial U(\rho x_0, x_1, x_2)}{\partial x_1}\, d\rho + W(x_1, x_2),$$

where the function $W(x_1, x_2)$ is chosen so that $\Delta_{(x_1,x_2)} W = \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\, \partial x_1}$, and

$$V_2(x) := \int_0^1 \left[- \begin{vmatrix} x_0 & x_2 \\ \frac{\partial U(tx)}{\partial x_0} & \frac{\partial U(tx)}{\partial x_2} \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ \frac{\partial V_1(tx)}{\partial x_1} & \frac{\partial V_1(tx)}{\partial x_2} \end{vmatrix} \right] dt.$$

Then the function $\mathbf{f} := U + \mathbf{i}\, V_1 + \mathbf{j}\, V_2$ is monogenic in B. Moreover, the most general monogenic function $\mathbf{g}$ having U as its scalar part is given by

$$\mathbf{g}(x) = \mathbf{f}(x) + \boldsymbol{\varphi}(x_1, x_2),$$

where $\boldsymbol{\varphi}(x_1, x_2)$ is a hyperholomorphic constant, that is, $D\boldsymbol{\varphi} = \overline{D}\boldsymbol{\varphi} = 0$.

ISOPERIMETRIC PARAMETER AS A SHAPE PARAMETER FOR BIOLOGICAL CELLS. NEW APPROACH

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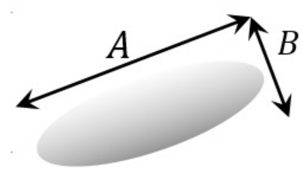
In biology , the deformation index is used as a parameter to describe the change in shape of biological cells. It is defined as

$$DI = \frac{A-B}{A+B}$$



where A is the length of the large axis of cell and B the small one. However, in many cases after cell deformation it is impossible to determine A and B. In this case, we propose to use the isoperimetric parameter as the cell shape parameter.

Shape parameter changes induced after ionizing radiation were studied using computational analysis. For our study, we used software Lab View permitting us to turn the source image obtained from polarized light microscope to input data for software "Nova". Obtained optical images were preliminarily analyzed using Nova's Section mode by filtering after magnification or analyzed the distortions of pixels. Grain Analysis mode represents a source image, a section of the source image, a table of geometrical parameters of bacterial cell (area, average size, perimeter, length, volume etc.) and a histogram of distribution density of one of the parameters of grains. For our study, we use only area, average size and perimeter parameters. Importing derived geometrical parameters into MS Excel, we determine shape parameter offered by us. Shape parameter is equal to the ratio of the perimeter to the square on 4×area (Eq. 1)

$$\alpha = \frac{P^2}{4S}. \qquad (1)$$

The purpose of this work was, by using the isoperimetric parameter as shape parameter of cell, to make computational analysis of ionizing radiation-induced shape parameter changes on bacterial cell. For our investigation we have radiated Escherichia coli and Pseudomonas aeruginosa. Strains of Escherichia coli were exposed to UV light for 10 and 15 minutes, Pseudomonas aeruginosa were exposed to gamma radiation for 15 and 30 minutes respectively. The results of radiation showed that gamma radiation causes decrease of area and perimeter and increase of shape parameter. Results for Escherichia coli were a little different after 10 minutes area and perimeter were decreased compared with control, shape parameter was increased but after 15 minutes area and perimeter were increased, shape parameter was decreased. This new approach makes it possible to use calculus of variaton methods in biology

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RELATIONS INVOLVING ZEROS AND SPECIAL VALUES OF THE ZETA AND ALLIED FUNCTIONS

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Some recurrence relations for the Riemann zeta function $\zeta(s)$ at integer arguments, as well as relations involving nontrivial zeros of $\zeta(s)$ have been derived in [1]; for instance,

Proposition 1 ([1]). *For the Riemann zeta function $\zeta(s)$, it holds that*

$$\zeta(k)+\sum_{j=1}^{k-2}\lambda_j\zeta(k-j)+\gamma\lambda_{k-1}+k\lambda_k=0,\quad (k\geq 2),$$

where λ_k are the coefficients in the Taylor series expansion of $\tilde{\Gamma}(z)=1/\Gamma(1-z)$ around $z=0$, and γ is the Euler-Mascheroni constant.

Proposition 2 ([1]). *For the nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$,*

$$\sum_{j=0}^{k-2}\lambda_j\left(\sum_{\rho}\frac{1}{\rho^{k-j}}\right)+\left[\frac{1}{2}\gamma+1-\log(2\sqrt{\pi})\right]\lambda_{k-1}+k\lambda_k=0,\quad (k\geq 2),$$

where λ_k are the coefficients in the Taylor series expansion of the Riemann ξ-function (or completed zeta function) around $s=0$, and γ is the Euler-Mascheroni constant.

The above propositions have been obtained using several formulas that were derived for the class of entire and meromorphic functions and that relate the sums of the nth powers of the reciprocals of zeros and poles of these functions with the coefficients of their Taylor series expansions [1], e.g.

Theorem ([1]). *Let $f(z)$ be an entire function of finite order ρ, for which $p=\lfloor\rho\rfloor$, where p is the genus of f. Further, suppose that $\{a_n\}_{n\geq 1}$ is the sequence of zeros of $f(z)$. Then*

$$\sum_{j=0}^{k-p-1}\lambda_j\sigma_{k-j}+k\lambda_k=\sum_{j=k-p}^{k-1}\lambda_j(k-j)q_{k-j},\quad k>p$$

where λ_j are the coefficients in the Taylor series expansion of $f(z)$ at $z = 0$, q_k are the coefficients of the polynomial in the Hadamard factorization of f, and σ_k are the sums of the form $\sigma_k = \sum_{n=1}^{\infty} 1/a_n^k$. Moreover, the above assertion holds true when ρ is not an integer.

The aim of this talk is to present other recurrence formulas involving Riemann zeta function and some other allied functions [2]. In particular, similar results are established for the digamma function, Barnes G-function (or double gamma function), and its logarithmic derivative [3], which will be based on the results of the paper [1] and the Weierstrass infinite product representations of entire functions [3, 4]. A recurrence formula for the values of the Riemann zeta function at odd positive integers is also derived. I will also discuss the relations for the multiple gamma functions [5] and Vignéras' multiple gamma functions [6]. The determinantal formulas for the sums of the nth powers of the reciprocals of zeros of entire or meromoprhic functions as well as for the coefficients λ_k will also be presented. I will conclude by discussing some further research in this direction and outlining possible applications of the same ideas and the results obtained.

Keywords: Zeta functions, sums of powers of reciprocals of zeros, entire functions, recurrence formulas

2020 Mathematics Subject Classification: 11M06, 30C15, 33B15, 30D99

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SOME RESULTS ON INTEGER PARTITIONS, COMPOSITIONS, AND THEIR GENERATING FUNCTIONS, WITH APPLICATIONS

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We begin a systematic study of compositions of n, with each part not exceeding m, denoted by $C(m,n)$, and the study of composition functions $c(m,n)$ [1]. We derive the recurrence formula, $c(m,n) = c(m,n-1) + c(m,n-m), n > m$, and also the generating function for $c(m,n)$, which coefficients are expressed by the formula $c(m,n+m) = \sum_{k=0}^{\lfloor n/m \rfloor} \binom{n-(m-1)k}{k}$. This formula is also obtained by combinatorial arguments. We aso study the functions given on a set of compositions.

Theorem 1. *Let $f(n) : \mathbb{N} \to R$, where $\mathbb{N}$ is a set of nonnegative integers and R is a ring, and let $F(q) = \sum_{n=0}^{\infty} f(n)q^n$ its generating function. Then for the function*

$$g(n) = \sum_{\pi \in C_n} f(\lambda_1)f(\lambda_2)\cdots f(\lambda_k), \quad g(0) = 1,$$

we have

$$(1) \quad g(n) = \sum_{k=0}^{n-1} f(k)g(n-1-k), \qquad (2) \quad G(q) = \frac{1}{1-qF(q)}.$$

We provide many examples illustrating the above theorem, and several sequences of [2] arise in this way. We also give the enumeration formulas for compositions with various restrictions on parts, and find several generating functions for Fibonacci sequences. By applying the formulas to linear recurrence relation of order k, we get the corresponding generating function, and in particular find the relation between the Pell numbers P_n and compositions of n. As corollaries we get recurrence relations for reciprocal generating functions, from which we derive the recurrence formulas for Bernoulli and Euler numbers. We also obtain a recurrence expression for generalized reciprocal generating function. For the partition function $p(m,n)$ we construct the generating function and find its relation with the Euler's formula, and we also propose a couple of conjectures. Interestingly, just a few of integer sequences described by the obtained class of generating functions of $c(m,n)$ appear to be associated with compositions in [2].

We also define an arbitrary order generalization of the Fibonacci and Lucas numbers [3]. By defining the roots of the associated auxiliary equation in terms of the roots of unity, summation over partitions generates results for generalized Fibonacci numbers and Lucas numbers, whereas summation over compositions leads to results for products of Fibonacci numbers related to Lucas numbers as well as integers in general.

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THE IDEA OF COMPLEX CONJUGACY
(an analogue of the idea of Riemann surfaces in differential geometry and real analysis)

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In short, the idea is to establish relationships between the surfaces M in R^3 and some complex functions $w(z)$ assoshiated with M; accordingly the idea is to apply complex functions $w(z)$ in studying surfaces M. In particular, real functions $u(x,y)$ also constitute a surface; thus the idea also applies to real functions.

In some ways, this idea is similar to the idea of Riemann surfaces.

Application of this idea leads to some results in differential geometry. In particular, analogues of the main theorems of classical Nevanlinna theory (in complex analysis) are obtained that are valid for generalized minimal surfaces (in geometry).

We also pose some problems that, in my opinion, can lead to a new crossroads between complex analysis, differential geometry and real analysis.

ABOUT A METABELIANITY OF CANONICAL QUOTIENT GROUP FOR GROUP OF LINE HOMEOMORPHISMS PRESERVING ORIENTATION

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At classification of groups of line homeomorphisms preserving orientation the most important characteristics are the metric invariants such as the invariant (0-projectively invariant), the projectively invariant (1-projectively invariant) and the ω-projectively invariant measures, where ω is cardinal number. The central object at the description of metric invariants is the minimal set because the support of a metric invariant is connected with him. The formulation of criteria of existence of metric invariants and their equivalent reformulations by topological, algebraic, combinatory terms and other characteristics allow to study a structure of groups with such invariants [4, 5]. In particular, the metabelianity of the quotient group G/H_G is established for group G with projectively invariant measure, where elements of normal subgroup H_G are homeomorphisms from group G for which each point of the minimal set is stationary.

Other task is the studying of structure of the concrete classes of groups of line homeomorphisms preserving orientation and, in particular, of the nilpotent groups. For the abstract finitely generated solvable groups the criterion of almost nilpotency is well-known [1]. For the finitely generated groups of interval diffeomorphisms preserving orientation and having high smoothness $G \subseteq Diff_+^{1+\alpha}, [0,1]), \quad \alpha > 0$ [2], for groups of line diffeomorphisms preserving orientation $G \subseteq Diff_+^1(\mathbb{R})$ with mutually transversal elements [6], as well as for groups of line homeomorphisms preserving orientation $G \subseteq Homeo_+(\mathbb{R})$ and satisfying to the maximality condition [7] the criteria of almost nilpotency were also established. In all three works the most difficult part is a proof of solvability of initial group. Further, the almost nilpotency of such group follows from the Rosenblatt's theorem. Therefore establishment of the solvability fact for groups of line homeomorphisms preserving orientation and also solvability of canonical quotient group is an important independent task.

In work [3] the important indication of existence of the projectively invariant Borel measure which is finite on compacts was received for groups of line homeomorphisms preserving orientation with freely acting element by terms of a finite normal row for which quotients are locally subexponential.

The new criterion of existence of the projectively invariant Borel measure which is finite on compacts is presented in the report for groups G of line homeomorphisms preserving orientation with a nonempty minimal set (in particular, with freely acting element). The formulation is given by terms of a quotient group G/H_G with a finite normal row in which quotient do not contain the free subsemigroups with two generators. Such criterion is equivalent to a metabelianity of the canonical quotient group G/H_G. It allows to give reformulations of criteria of the invariant and the projectively invariant Borel measures which are finite on compacts by terms of a chain of the enclosure of the classes of quotient groups G/H_G. It is shown that in space of quotient groups G/H_G for groups of line homeomorphisms preserving orientation $G \subseteq Homeo_+(\mathbb{R})$ with a nonempty minimal set the class of metabelian groups coincides with of groups class with a finite normal row with quotients not containing the free subsemigroups with two generators and the class of commutative groups coincides with of groups class not containing the free subsemigroups with two generators. Combinatory complexity was established for groups of line homeomorphisms preserving orientation $G \subseteq Homeo_+(\mathbb{R})$ with a nonempty minimal set, with not trivial quotient group $G/H_G \neq < e >$ and without freely acting homeomorphism. Such group is not group with a finite normal row with quotients not containing free subsemigroups with two generators, in particular, such group contain free subsemigroups with two generators.

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ON EQUIVALENCE RELATION OF CRYPTOGRAPHIC FUNCTIONS

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We will consider modern applications of Boolean functions:

- reliability theory, multicriteria analysis, mathematical biology, image processing, theoretical physics, statistics;
- voting games, artificial intelligence, management science, digital electronics, propositional logic;
- algebra, projective geometry, coding theory, combinatorics, sequence design, cryptography;
- cryptographic attacks on block ciphers and corresponding properties of S-boxes.

FIRST-ORDER MODEL THEORY, SURJUNCTIVITY, AND KAPLANSKY'S STABLE FINITENESS CONJECTURE

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A ring R is directly finite if $ab = 1$ implies $ba = 1$ for all $a, binR$. A ring R is stably finite if the ring $M(R, d)$ of d-by-d matrices with entries in R is stably finite for every integer $d \geq 1$. A group G is surjunctive if for any finite alphabet set A, every injective cellular automaton (i.e., injective, continuous, G-equivariant map) $T\colon A^G \longrightarrow A^G$ is surjective. Using algebraic geometry methods, Xuan Kien Phung proved that the group ring $K[G]$ of a surjunctive group G with coefficients in a field K is always stably finite. In other words, every group satisfying "Gottschalk's conjecture" also satisfies "Kaplansky's stable finiteness conjecture". Based on a joint work with Michel Coornaert and Phung, I'll present a proof of this result based on first-order Model Theory.

ON SOME QUANTIFIED PROPOSITIONAL SYSTEM

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In this paper some new quantified propositional proof system is introduced and compared by proof steps with the other quantified and not quantified propositional proof systems.

Introduction. Propositional proof complexity has its origin in the seminal paper by Cook and Reckhow [1]. It provides a path for approaching the P vs. NP problem: proving super-polynomial lower bounds to all propositional proof systems is equivalent to showing that NP is different from coNP and therefore P is different from NP. It is well known that the exponential lower bounds for proof sizes of some sets of tautologies are obtained in many systems, but for some most natural calculi, in particular for Frege systems, the question about polynomial bounded sizes is still open. While traditionally the complexity of propositional tautologies proofs has been at the centre of research, the past two decades have witnessed a surge in proof complexity of quantified boolean formulas (QBFs), which give not only a new class of tautologies, but in some of quantified systems quantifier-free tautologies can be proved simplier. Some interesting survey of proof complexity for QBFs is given in [2], where the complexities for three QBF families are compared in different quantified propositional proof systems: variants of QBF resolution, QBF Frege systems, quantified version of cutting planes, QBF sequent calculi and some others.

On the base of propositional system GS (Generalized Splitting), described in [3], a new quantified propositional proof system is introduced here. The place of the system GS in the hierarchy of the propositional proof systems [1] is still unknown and moreover: by the comparison of the two main proof complexity characteristics (*steps* and *size*) for two classes of formulas in the system GS and Frege systems it is shown that for one class of considered formulas the bounds in the system GS are much better, than in Frege systems and for the second class – quite the reverse [4]. For all above mentioned it is follow, that the investigations of proof complexities in some quantified variant of the system GS are also important.

Preliminaries. We will use the current concepts of a propositional formula, a proof system for propositional logic, proof complexity, and well known notions of polynomially equivalence and exponential speed-up. The language of considered systems contains the propositional variables, logical connectives $\neg$, &, $\vee$, $\supset$ and parentheses (,). Following the usual terminology we call the variables and negated variables *literals*. In [3] the following notions were introduced. We call a *replacement-rule* each of the following trivial identities for a propositional formula ψ.

$$\begin{array}{rrrr}
0\&\psi = 0; & \psi\&0 = 0; & 1\&\psi = \psi; & \psi\&1 = \psi; \\
\psi\&\neg\psi = 0; & \neg\psi\&\psi = 0; & \psi\&\psi = \psi; & \\
0 \vee \psi = \psi; & \psi \vee 0 = \psi; & 1 \vee \psi = 1; & \psi \vee 1 = 1; \\
\psi \vee \neg\psi = 1; & \neg\psi \vee \psi = 1; & \psi \vee \psi = \psi; & \\
0 \supset \psi = 1; & \psi \supset 0 = \neg\psi; & 1 \supset \psi = \psi; & \psi \supset 1 = 1; \\
\psi \supset \neg\psi = \neg\psi; & \neg\psi \supset \psi = \psi; & \psi \supset \psi = 1; & \\
\neg 0 = 1 & \neg 1 = 0; & \neg\neg\psi = \psi. &
\end{array}$$

Application of a replacement-rule to some word consists in the replacing of some its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right- hand side

The proof system GS. Let φ be some formula and p be some of its variable. Results of splitting method of formula φ by variable p (splinted variable) are the formulas $\varphi[p^{\delta}]$ for every δ from the set $\{0,1\}$, which are obtained from φ by assigning δ to each occurrence of p and successively using replacement-rules. The generalization of splitting method allow as associate with every formula φ some tree with root, nodes of which are labeled by formulas and edges, labeled by literals. The root is labeled by itself formula φ. If some node is labeled by formula v and α is some its variable, then both edges, which going out from this node, are labeled by one of literals α^{δ} for every δ from the set $\{0,1\}$, and every of 2 "sons" of this node is labeled by corresponding formula $v[\alpha^{\delta}]$. Each of the tree's leafs is labeled with some constant from the set $\{0,1\}$. The tree, which is constructed for formula φ by described method, we will call ***splitting tree*** (s.t.) of φ. It is obvious, that changing the order of splinted variables in given formula φ, we can obtain the different splitting trees of φ.

The **GS** proof system can be defined as follows: for every formula φ must be constructed some s.t. and if all tree's leafs are labeled by the value 1, then formula φ is tautology and therefore we can consider the pointed constant 1 as axiom, and for every formula v, which is label of some s.t. node, and p is its splinted variable, then the following figure $v[p^0], v[p^1] \vdash v$ can be considered as some inference rule, hence every above described s.t. can be consider as some proof of φ in the system **GS**. The complexity of s.t. is the *number of different formulas*, with which labeled its nodes. **The proof complexity of tautology φ in the system GS** is value of minimal complexity of its splitting trees.

Note that if we consider splitting method for formulas given in disjunctive normal form, then GS system is the well-known system Analytic Tableaux.

Main results. Quantified Splitting system (QS). A QBF is a propositional formula augmented with Boolean quantifiers $\forall$, $\exists$ that range over the Boolean values 0, 1. Every propositional formula is already a QBF. Let ϕ be a QBF. The semantics of the quantifiers are that: $\forall x\phi(x) \equiv \phi[x^0]\&\phi[x^1]$ and $\exists x\phi(x) \equiv \phi[x^0] \vee \phi[x^1]$. In computer science investigated standardised QBF, all quantifiers appear outermost in a (quantifier) prefix, and are followed by a propositional formula, called the *matrix*. The variables, following after quantifier $\forall$ called *universal* variables and the variables, following after quantifier $\exists$ called existential *variables*. The system **QS** works as follows: for any QBF formula φ we use the system **GS** to matrix of φ. S.t. for every GBF tautology φ must be the following: if for any step the splinted variable α is universal variable of φ, then both subtrees, stuffed from the α^0 and α^1 labeled edges must have some branch, ended with value 1 labeled leafs; if for any step the splinted variable α is existential variable of φ, then at least one of subtrees, stuffed from the α^0 or α^1 labeled edges must have some branch, ended with value 1 labeled leafs.

Theorem. 1) *The systems GS and QS are polinomially equivalent by steps,*
2) *The system QS has exponentially speed-up by steps over the variants of QBF resolution system.*

Proof of 1) is based on the good choices of splinted variables sentences.
Proof of 2) is based on the investigation of proof steps of equality families of

$$SC_n = \exists \boldsymbol{x_1},\ldots,\boldsymbol{x_n} \forall \boldsymbol{u_1},\ldots,\boldsymbol{u_n} \exists \boldsymbol{t_1},\ldots,\boldsymbol{t_n} \left(\bigwedge_{1\le i\le n} (\boldsymbol{x_i} \Leftrightarrow \boldsymbol{u_i}) \rightarrow \bar{\boldsymbol{t}}_{\boldsymbol{i}} \right) \wedge \left(\bigvee_{1\le i\le n} \boldsymbol{t_i} \right)$$

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i-QUASIGROUP AND LEFT BOL QUASIGROUP

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Florya I.A. in [1] defined and researched left Bol quasigroup. In [2], together with Didurik N.N., we defined and researched i-quasigroups. We find a condition when any i-quasigroup is a left Bol quasigroup.

Definition 1. *Quasigroup* $(Q,\cdot)$ *with identity:*

$$x(y\cdot xz) = R_{e_x}^{-1}(x\cdot yx)\cdot z, \quad \forall x,y\in Q, \tag{1}$$

where $R_{e_x}y = y\cdot e_x$, $xe_x = x$, *is called left Bol quasigroup [1].*

Definition 2. *Quasigroup* $(Q,\cdot)$ *with i-identity*

$$x(xy\cdot z) = y(zx\cdot x), \quad \forall x,y,z\in Q \tag{2}$$

is called i-quasigroup.

Theorem. *Any i-quasigroup* $(Q,\cdot)$ *with left unit* f, *where* $f\cdot x = x, \forall x\in Q$, *is a left Bol quasigroup, i.e., identity* (1) *is satisfied in* $(Q,\cdot)$.

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ON SEMISYMMETRIC T-QUASIGROUPS

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We use [1,2].

Definition 1. *A quasigroup $(K,\cdot)$ is called semisymmetric, if the semisymmetric law is satisfied in $(K,\cdot)$, $x\cdot yx = y$, $\forall x,y\in K$ [1, 2].*

Definition 2. *Quasigroup $(Q,\cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q,+)$, its automorphisms φ and ψ and a fixed element $a\in Q$ such that $x\cdot y = \varphi x+\psi y + a$ for all $x,y\in Q$. A T-quasigroup with the additional condition $\varphi\psi=\psi\varphi$ is medial [3].*

Theorem. *In T-quasigroup $(Q,\cdot)$ of the form $x\cdot y=\varphi x+\psi y$ semisymmetric identity $(x\cdot yx = y)$ is true if and only if $\psi=\varphi^{-1}$, $\psi^3 = I$, where $x + Ix = 0$ $\forall x,y\in Q$.*

Example. 1. The cyclic group Z_2 satisfies semisymmetric identity. Indeed, we can use commutative and associative identities that are true in the group Z_2.

2. We will notice that $AutZ_m \cong Z_m^*$ [4, p.61], $AutZ_7\cong Z_7^*\cong Z_6$. We suppose that $(Z_7,\circ)$, $x\circ y=\psi^{-1}x+\psi y$, where $(Z_7,+)$ is the cyclic group of order 7, $\psi^6=\varepsilon$, $\psi^3 = I$.

+	0	1	2	3	4	5	6	∘	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6	0	0	5	3	1	6	4	2
1	1	2	3	4	5	6	0	1	3	1	6	4	2	0	5
2	2	3	4	5	6	0	1	2	6	4	2	0	5	3	1
3	3	4	5	6	0	1	2	3	2	0	5	3	1	6	4
4	4	5	6	0	1	2	3	4	5	3	1	6	4	2	0
5	5	6	0	1	2	3	4	5	1	6	4	2	0	5	3
6	6	0	1	2	3	4	5	6	4	2	0	5	3	1	6

Here $I = (0)(16)(25)(34)$, $\psi = (0)(154623)$, $\psi^{-1} = (0)(132645)$ $\psi^6 = \varepsilon$, $\psi^3 = I$.

3. Direct product of finite number of the group Z_2 and quasigroup $(Z_7, \circ)$ defined in item 2.

Notice, using Mace [5] we construct semisymmetric non-medial quasigroups. See below.

*	0	1	2	3	4	5	6
0	0	1	2	4	5	3	6
1	1	0	3	2	4	6	5
2	2	3	0	1	6	5	4
3	5	2	1	6	0	4	3
4	3	4	6	5	1	0	2
5	4	6	5	0	3	2	1
6	6	5	4	3	2	1	0

$12 * 34 = 3 * 0 = 5$, $13 * 24 = 2 * 6 = 4$.

*:	0	1	2	3	4
0	2	1	0	3	4
1	1	0	3	4	2
2	0	4	2	1	3
3	3	2	4	0	1
4	4	3	1	2	0

$02 * 34 = 0 * 1 = 1$, $03 * 24 = 3 * 3 = 0$.

*:	0	1	2	3	4	5
0	2	1	0	5	3	4
1	1	0	3	4	2	5
2	0	4	2	1	5	3
3	4	2	5	3	1	0
4	5	3	1	0	4	2
5	3	5	4	2	0	1

$12 * 30 = 3 * 4 = 1$, $13 * 20 = 4 * 0 = 5$.

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DOPPELSEMIGROUPS AND THEIR UPFAMILY EXTENSIONS

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In the talk we shall discuss the algebraic structure of doppelsemigroups and their upfamily extensions. By definition, a *doppelsemigroup* is an algebraic structure $(D,\dashv,\vdash)$ consisting of a non-empty set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(D_1)\ \ (x \dashv y) \vdash z = x \dashv (y \vdash z),$$

$$(D_2)\ \ (x \vdash y) \dashv z = x \vdash (y \dashv z).$$

If $(D,\dashv,\vdash)$ is a doppelsemigroup, then rearranging the parentheses in an expression that contains only operations $\vdash$, $\dashv$ and elements of D do not change the result. A doppelsemigroup $(D,\dashv,\vdash)$ is called *commutative* [6] if both semigroups $(D,\dashv)$ and $(D,\vdash)$ are commutative. A doppelsemigroup $(D,\dashv,\vdash)$ is said to be *strong* [7] if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z).$$

In [2], the task of describing all pairwise non-isomorphic (strong) doppelsemigroups with at most three elements has been solved. We proved that there exist 8 pairwise non-isomorphic two-element doppelsemigroups among which 6 doppelsemigroups are commutative. All two-element doppelsemigroups are strong. It was proved that there exist 75 pairwise non-isomorphic three-element doppelsemigroups among which 41 doppelsemigroups are commutative. Non-commutative doppelsemigroups are divided into 17 pairs of dual doppelsemigroups. Also up to isomorphism there are 65 strong doppelsemigroups of order 3, and all non-strong doppelsemigroups are not commutative. In [3], we studied cyclic doppelsemigroups. A doppelsemigroup $(G,\dashv,\vdash)$ is called *a group doppelsemigroup* if $(G,\dashv)$ is a group. A group doppelsemigroup $(G,\dashv,\vdash)$ is said to be *cyclic* if $(G,\dashv)$ is a cyclic group. It was proved that up to isomorphism there exist $\tau(n)$ finite cyclic (strong) doppelsemigroups of order n, where τ is the number of divisors function. There exist infinite many pairwise non-isomorphic infinite cyclic (strong) doppelsemigroups.

A family $\mathcal{M}$ of non-empty subsets of a set X is called *an upfamily* if for each set $A \in \mathcal{M}$ any subset $B \supset A$ of X belongs to $\mathcal{M}$. By $\upsilon(X)$ we denote the set of all upfamilies on a set X. Each family $\mathcal{B}$ of non-empty subsets of X generates the upfamily $\langle \mathcal{B} \rangle = \{A \subset X : \exists B \in \mathcal{B}\ (B \subset A)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called the *Stone-Čech compactification* of X, see [5]. An ultrafilter $\langle\{x\}\rangle$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle\{x\}\rangle$ generated by the singleton $\{x\}$, and hence we can consider $X \subset \beta(X) \subset \upsilon(X)$. It was shown in [1] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $* : \upsilon(S) \times \upsilon(S) \to \upsilon(S)$ by the formula

$$\mathcal{L} * \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle$$

for upfamilies $\mathcal{L}, \mathcal{M} \in \upsilon(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $\upsilon(S)$.

In [4], it was shown that the upfamily extension $(\upsilon(D), \dashv, \vdash)$ of a (strong) doppelsemigroup $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup as well. Also we introduced the upfamily functor υ in the category **DSG** of doppelsemigroups and their homomorphisms and showed that this functor preserves strong doppelsemigroups, doppelsemigroups with left (right) zero, doppelsemigroups with left (right) identity, left (right) zeros doppelsemigroups. On the other hand, the functor υ does not preserve commutative doppelsemigroups and group doppelsemigroups. It was proved that the automorphism group of the upfamily extension of a doppelsemigroup $(D, \dashv, \vdash)$ of order $|D| \geq 2$ contains a subgroup, isomorphic to $C_2 \times \mathrm{Aut}\,(D, \dashv, \vdash)$. Also we described the structure of upfamily extensions of all two-element doppelsemigroups and their automorphism groups.

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CONTINUOUS GROUPS OF LEFT QUASIGROUP OPERATIONS

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Representations of groups by invertible binary operations of a set play an important role in algebra and other branches of mathematics. One of the old problems of algebra, the problem of describing those groups that can be multiplicative groups of fields, was solved in terms of binary representations of groups by Yu. Movsisyan [1].

The study of binary representations of a topological group G or binary G-spaces was started in [2]. In a broader sense, a binary representation of a topological group G should be understood as a homomorphism of G into the group of all invertible continuous binary operations or left quasigroup operations of a topological space X. This concept in algebra was considered and studied in [3].

Some important notions and results of the theory of binary G-spaces are considered.

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A MODEL OF EXACTLY CONSTRUCTIBLE QUANTUM THERMO-DYNAMICS FROM FIRST PRINCIPLES

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A microscopic theory of quantum thermodynamics has been developed for a multicomponent system of reacting particles, one of the components of which is a subsystem of $4D$ isotropic quantum oscillators - a test sub-system. The medium is considered as a test sub- system consisting of single-particle systems immersed in a random environment, which, as a result of evolution in the limit of thermodynamic equilibrium, with different probabilities pass into topologically different final quantum states. Within the framework of a stochastic differential equation (SDE) of the Langevin-Schrödinger (L-Sch) type, which describes the motion of a test particle in a reacting medium, the problem of self-organization of a single-particle system with its environment is investigated. By the help of a low-dimensional reference SDE, the original L-Sch equation is reduced to an autonomous form, which is then solved explicitly as an orthogonal basis of random processes in Hilbert space. Assuming that the interaction of a single-particle system with the environment is described by a complex Gauss-Markovian random processes, taking into account the reference SDE, a Fokker-Planck (F-P) type equation for the distribution of environmental fields is derived. Using the F-P equation, the measure of the functional space and, accordingly, the mathematical expectation of the time-dependent wave function of a single-particle system are constructed. In the limit of statistical equilibrium, the time-dependent entropy and complexity of a single-particle system are studied in detail, taking into account the influence of its environment. It is shown that all thermodynamic potentials of a statistical ensemble can be constructed in the form of functional integrals, which are then calculated exactly using the generalized Feynman-Kac theorem and reduced to double-integral representations with solutions of second-order partial differential equations.

It is proved that when imposing an additional constraint on the wave function of a $4D$ isotropic oscillator, the representation describes the quantum thermodynamics of a multicomponent ensemble of particles, the test-particle of which is a reacting hydrogen atom. The developed model of quantum thermodynamics is also interesting in that there are no restrictions on the power of interaction with the environment, which makes the approach suitable for studying atomic-molecular processes far from the state of thermodynamic equilibrium of the environment, in critical states, when elementary processes occur under the strong influence of collective effects.

ON THE GENERATION OF BELL STATES USING RANDOM MAPPINGS OF THE FOCK BASIS

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Entanglement of the spatial degrees of freedom of photons is one of the most important directions for the implementation of quantum communications. There are various rather complex and expensive experiments on the generation of entangled photons, in which the environmental noise is the main factor in the decoherence of these states. We propose to generate Bell states by passing photons through an optical waveguide with random boundaries, realizing both elastic and inelastic scattering of photons in the waveguide. Obviously, photon entanglement is a purely quantum phenomenon, mathematically described by Fock states. Mathematically, the problem is as follows: initially we have a one-particle $2D$ Fock state, carry out random mappings over this state and, in the limit of statistical equilibrium, find the mathematical expectation of the wave function of two $1D$ entangled Fock states.

In particular, using the obvious similarity of the neutrino and the photon, we proved that the propagation of a photon in a $3D$-inhomogeneous medium can be described by complex probabilistic processes satisfying the Lagevin-Weyl type equation:

$$\partial_t \Psi_\pm(\mathbf{r},t) \pm c(\mathbf{r})\Big(\mathbf{S}\cdot\nabla\Big)\Psi_\pm(\mathbf{r},t) = 0, \qquad \partial_t \equiv \partial/\partial t, \tag{1}$$

where $c(\mathbf{r})$ denotes the speed of light in $3D$ Euclidean space $\mathbf{r} \equiv \mathbf{r}(x,g,z) \in \mathbb{R}^3$, in addition, Ψ_+ and Ψ_- denote the wave functions of photons of both helicities, a left-hand +1 and right-hand -1, respectively. In Eq.s (1) the set of matrix $S = (S_x, S_y, S_z)$ have the form:

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the system of first-order differential equations (1), we find the system of second-order equations by differentiating with respect to time:

$$\partial_t^2 \Psi_\pm(\mathbf{r},t) - c(\mathbf{r})\Big(\nabla c(\mathbf{r})\nabla\Big)\Psi_\pm(\mathbf{r},t) + c^2(\mathbf{r})\nabla\cdot\Big(\nabla\Psi_\pm(\mathbf{r},t)\Big) = 0. \tag{2}$$

Assuming that the z axis coincides with the direction of photon propagation, and considering that in this case $\Psi_\pm(\mathbf{r},t) = (\Psi_\pm^x, \Psi_\pm^y, \Psi_\pm^z \equiv 0)$, from the equations' system (2) we can find the following second order partial differential equations (PDEs):

$$\begin{aligned}
&\frac{\partial^2\Psi_\pm^x}{\partial t^2} - c^2\left\{\frac{\partial^2\Psi_\pm^x}{\partial x^2} + \frac{\partial^2\Psi_\pm^x}{\partial y^2} + \frac{\partial^2\Psi_\pm^x}{\partial z^2}\right\} - cc_x\frac{\partial\Psi_\pm^x}{\partial x} = 0,\\
&\frac{\partial^2\Psi_\pm^y}{\partial t^2} - c^2\left\{\frac{\partial^2\Psi_\pm^y}{\partial x^2} + \frac{\partial^2\Psi_\pm^y}{\partial y^2} + \frac{\partial^2\Psi_\pm^y}{\partial z^2}\right\} - cc_y\frac{\partial\Psi_\pm^y}{\partial y} = 0,
\end{aligned} \tag{3}$$

where the following notations are made $c_\eta = \partial_\eta c$ and $\eta = x, y$.
Adding the equations (3) and assuming that $c_x = c_y = 0$, we find:

$$\frac{\partial^2\Psi_0}{\partial t^2} - c^2\Delta\Psi_0 = 0, \tag{4}$$

where $\Psi_0(x,y,z;t) = \Psi_\pm^x(x,y,z;t) + \Psi_\pm^y(x,y,z;t)$.
Now substituting the wavefunction of the photon in the form:

$$\Psi_0(x,y,z;t) = e^{i\omega_0 t + ik_z z}\mathcal{F}(x,y), \tag{5}$$

where ω_0 and k_z denote the frequency and momentum of the photon in the (in) vacuum state) from the equation in (4) one can get the following parabolic equation for evolution of a photon:

$$i\frac{\partial\mathcal{F}}{\partial\tau} + \frac{1}{2}\left\{\frac{\partial^2\mathcal{F}}{\partial x^2} + \frac{\partial^2\mathcal{F}}{\partial y^2}\right\} + \left(\frac{\omega_0^2}{c^2} - k_z^2\right)\mathcal{F} = 0, \quad \tau = \frac{z}{k_z}. \tag{6}$$

where $\tau = z/k_z$ plays the role of time.

Expanding into a series the speed of light $c(x,y)$ near the z axis, from equation (6) we find:

$$i\frac{\partial\mathcal{F}}{\partial\tau} + \frac{1}{2}\left\{\frac{\partial^2\mathcal{F}}{\partial x^2} + \frac{\partial^2\mathcal{F}}{\partial y^2}\right\} - \frac{1}{2}\Big\{A_1(\tau)x^2 + A_2(\tau)y^2 + 2A_3(\tau)xy\Big\}\mathcal{F} = 0, \tag{7}$$

where

$$A_1(\tau) = \frac{\omega_0^2}{c_0^2}c_{xx}(0,0,z),\ A_2(\tau) = \frac{\omega_0^2}{c_0^2}c_{yy}(0,0,z),\ A_3(\tau) = \frac{\omega_0^2}{c_0^2}c_{xy}(0,0,z).$$

Note that when deriving the equation (7) we assumed that $\omega_0^2 c_0^{-2} - k_z^2 = 0$, where $c_0(z) = c(z,0,0)$. In the case where the coefficients satisfy the inequalities $A_1(\tau) > 0$, $A_2(\tau) > 0$ and $A_3(\tau) > 0$ equation (7) describes the problem of a $2D$ quantum harmonic oscillator.

By using the coordinate transformations:

$$q_1 = (x-y)/\sqrt{2}, \qquad q_2 = (x+y)/\sqrt{2}, \tag{8}$$

we can diagonalize the equation (7) assuming $c_{xx}(0,0,z) = c_{yy}(0,0,z)$ and reduce it to the form:

$$i\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{2}\sum_{l=1}^{2}\Big[-\frac{\partial^2}{\partial q_l^2} + \Omega_l^2(\tau)q_l^2\Big]\mathcal{F}, \tag{9}$$

where $\Omega_l^2(\tau) = \Omega^2(\tau) - (-1)^l\Theta(\tau)$, $(l = 1,2)$ the square of frequency and the following notations are made:

$$\Omega_l^2(\tau) = \Big(\frac{\omega_0}{c_0}\Big)^2 c_{xx} = \Big(\frac{\omega_0}{c_0}\Big)^2 c_{yy}(\tau), \qquad \Theta(\tau) = \Big(\frac{\omega_0}{c_0}\Big)^2 c_{xy}(\tau).$$

We consider the frequency $\Omega_l^2(\tau)$ as the sum of regular and stochastic functions:

$$\Omega_l^2(\tau) = \Omega_{0l}^2(\tau) + f_l(\tau), \tag{10}$$

where $\Omega_{0l}(\tau)$ is a regular function and $f_l(\tau)$ is a complex random process satisfying the following asymptotic conditions:

$$\lim_{\tau\to\mp\infty}\Omega_{0l}(\tau) = \Omega_l^{\mp} > 0, \qquad \lim_{\tau\to-\infty} f_l(\tau) = 0. \tag{11}$$

Let us assume that $f_l(\tau) = f_l^{(r)}(\tau) + i f_l^{(i)}(\tau)$ is an independent Gaussian-Markov process with zero mean and delta-shaped correlation function:

$$< f_l^{(v)}(\tau) >= 0, \quad < f_l^{(v)}(\tau) f_l^{(v)}(\tau') >= 2\epsilon_l^{(v)}\delta(\tau-\tau'), \quad v = r,i, \tag{12}$$

where $\epsilon_l^{(r)}$ and $\epsilon_l^{(i)}$ denote the powers of elastic and inelastic collisions.

Thus, we have reduced the original problem of the propagation of a single photon along a waveguide with random boundaries to the problem of two coupled $1D$ quantum oscillators immersed in a random environment [1]. As shown in this paper, all parameters of the dynamical system under consideration can be constructed exactly in the form of double integral representations and reference equations (second-order partial differential equations with two variables).

In the paper, the mathematical expectation of the photon wave function is constructed in the limit of statistical equilibrium. The probabilities of the formation of various Bell states as a result of the decay of a single two-dimensional photon into two one-dimensional entangled photons have been studied in detail. A mathematical algorithm has been developed for the numerical solution of the problem, which is very important for the control and optimization of the corresponding Bell state.

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ENDOMORPHISM KERNEL PROPERTY FOR FINITE GROUPS

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The concept of the endomorphism kernel property for an universal algebra has been introduced by Blyth, Silva in [1] as follows.

An algebra A has the *endomorphism kernel property* (EKP) if every congruence relation θ on A different from the universal congruence $\iota_A = A \times A$ is the kernel of an endomorphism on A.

This concept was studied by many authors mainly for classes of universal algebras with (semi)lattice reduct. First attempt to describe one of "classical" algebraic structures – finite abelian groups – which posses (somewhat stronger property) strong endormorhism kernel property was done in 2020 by J. Fang and Z.-J. Sun in [2].

EKP is in the context of groups equivalent to the fact, that a group G has EKP if and only if for any normal subgroup H of G ($H \triangleleft G$) the factorization G/H is isomorhic to a subgroup of G.

Using the fact that a finite abelian group G is a direct product (sum) of its Sylow subgroups (which is true also for every nilpotent group) we prove

Theorem 1. *Let G be finite abelian group. Then G has EKP.*

This is not true for infinite abelian groups, for example the group $(\mathbb{Z}, +)$ does not have EKP, because (for example) the factorization $\mathbb{Z}/2\mathbb{Z}$ has 2 elements, but $\mathbb{Z}$ does not have 2 element subgroup.

In the case of non-abelian groups we were able to prove

Lemma. *Let p be a prime number, G be a non-abelian group, $|G| = p^3$.*

1. *If $p > 2$, then G has EKP.*
2. *If $p = 2$ and $G \cong D_4$, then G has EKP.*
3. *Let P be a non-abelian group, $|P| = p^3$ for an odd prime number p or $P = D_4$. Let $G = Z_p^k \times P$. Then G has EKP.*

Using this result and the structure of finite nilpotent groups we get

Theorem 2. *Let G be a finite nilpotent group written in the form $G = G_1 \times G_2 \times \cdots \times G_k$, where $G_i, i = 1, \ldots, k$ are Sylow p_i subgroups of G. Let each Sylow subgroup G_i be (isomorphic to) one of the following groups:*

1. *an abelian group,*
2. $Z_{p_i}^{k_i} \times P_i$, *where $k_i \geq 0$, $p_i > 2$ and P_i is a non-abelian group of order p_i^3,*
3. $Z_2^{k_i} \times D_4$, *where $k_i \geq 0$ and D_4 is a dihedral $8-$element group.*

Then G has EKP.

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BINARY REPRESENTATION OF MULTIPLICATIVE MONOID OF THE SEMIRING

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The aim of this talk is to introduce main concepts of theory for semirings and binary representations for defining the postulate of prime property for semirings. The main topic is the binary representation for multiplicative monoid of the semiring.

The following definitions will be given for more pressingly to introduce the main algebraic structures and methods. For first, there are definitions about semirings and their prime properties. These definitions can be found in [1, 6].

Definition 1. *The $A(+,\cdot,0,1)$ algebra with two binary and two unary operations is called semiring if:*

- *The $A(+,0)$ additive algebra is a commutative monoid.*
- *The $A(\cdot,1)$ multiplicative algebra is a monoid.*
- *The left and right distributive laws are true in A:*

$$x(y+z) = xy + xz,$$

$$(x+y)z = xz + yz.$$

- *And the additive unit is an absorbing element for multiplicative operation:*

$$x0 = 0x = 0.$$

Definition 2. *The semiring is called prime if the following identity is true on it:*

$$x + 1 = 1.$$

The definitions above show that the every associative ring is a definitely semiring, but not every semiring is an associative ring.

Main mathematical theory for this paper is the binary representations. The following definitions are giving the quick and short information, for more about this theory (containing the theorems and their proofs) can be found in [2–5].

Definition 3. *The* $\Gamma(\Sigma)$ *algebra can be represented with* $\mathcal{F}_G^2(\oplus, \circ)$ *binary operations algebra of the some* G *set where:*

- $G \cap \Gamma = \emptyset$
- $\mathcal{F}_G^2 = \{f \mid f\colon G \times G \to G\}$
- *The* $(\oplus)$ *and* $(\circ)$ *operations is defined by the following identities:*

$$(f \oplus g)(x,y) = f(x, g(x,y))$$

$$(f \circ g)(x,y) = f(g(x,y), y)$$

if there exists a homomorphism from Γ *to* $\mathcal{F}_G^2$*:*

$$\varphi\colon \Gamma \to \mathcal{F}_G^2.$$

Definition 4. *A binary representation is called* ***strict*** *if the homomorphism defining it is a monomorphism.*

The operations on $\mathcal{F}_G^2(\oplus, \circ)$ binary operations' algebra are associative and for each of them there exists the unit:

$$E \oplus f = f \oplus E = f,$$

$$F \circ f = f \circ F = f,$$

where:

$$E(x,y) = y, \quad F(x,y) = x.$$

All proofs for definition given above can be found in [3].

And the following theorem is describing the main result for paper which should be proven.

Theorem. *The* $A(+,\cdot,0,1)$ *semiring is a prime iff it's multiplicative monoid* $A(\cdot,1)$ *has strict binary representation given by the following mapping:*

$$\varphi\colon A(\cdot,1) \to \mathcal{F}_A^2(\oplus, E)$$

where

$$\mathcal{F}_A^2 = \{f_a \mid f_a(x,y) = x + ay,\ \forall a \in A\}.$$

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COMBINATORIAL INDICES OF GRAPHS AND MATRICES

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This talk is based on the joint works with Yu. A. Alpin, A. M. Maksaev, E. R. Shafeev.

Scrambling index is a fundamental invariant in graph theory and in the theory of non-negative matrices and their applications. Shortly, a scrambling index of a primitive directed graph G is the smallest positive integer $k = k(G)$ such that for any pair of vertices u, v of G there exists a vertex w of G such that there are directed walks of length k from u to w and from v to w. If a digraph G is not primitive, then it can appear that the integer k described above does not exist. In this case we say that $k(G) = 0$, otherwise we define $k(G)$ as in the primitive case.

The scrambling index is important for several applications. In particular, if A is an $n \times n$ non-negative primitive stochastic matrix with a non-unit eigenvalue λ, and k is the scrambling index of $G(A)$, then $|\lambda| \leq (\tau_1(A^k))^{1/k} < 1$, where τ_1 is a certain matrix invariant, usually called Dobrushin coefficient.

Also scrambling index provides lower bounds for the length of reset words for synchronizing automata, since it gives a lower bound for the exponent of the graph representing this automata.

More applications are in the theory of memoryless communication systems and related areas. Scrambling index for primitive graphs was an object of intensive investigations starting from the works by Seneta, Paz, Akelbek, Kirkland, and others.

We prove that for non-primitive digraphs on n vertices the following bound for scrambling index is true $k(G) \leq 1 + \left\lceil \frac{(n-2)^2+1}{2} \right\rceil$. We characterize such graphs with the maximal scrambling index and characterize non-primitive graphs possessing positive scrambling index.

Several generalizations of scrambling index and their bounds will be discussed in the talk. As a corollary we obtain a generalization of the theorem by Protasov and Voinov about the combinatorial structure of semigroups of nonnegative matrices, for not necessarily irreducible semigroups of matrices. For this purpose, an extensions of the concepts of imprimitivity index and canonical partition are introduced which are based on the chain properties of non-negative matrices.

In addition we investigate linear transformations preserving scrambling index of graphs. The theory of transformations preserving different matrix properties and invariants dates back to the works of Frobenius, Schur, and Dieudonne and is an intensively developing part of linear algebra and its applications nowdays. In particular, it is natural to investigate these transformations for combinatorial or graph theory invariants. We show that linear transformations preserving scrambling index of graphs are always bijective, which is not the case for cyclicity index, for example. The structure of linear maps preserving only several values of the scrambling index are characterized. We also discuss linear maps preserving other matrix indices mentioned above.

CENTERS AND MEDIANS OF TREES

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Many practical problems in operations research, computational mathematics, automatic planning, and control theory are designed to address object-related problems. Among such tasks, a special place is occupied by the study of the central regions of these objects. We study the relationships between centers and medians graphs, which are trees.

Let $G = (V, E)$ be a connected undirected graph, where V is the set of vertices and E is the set of edges. Let us give definitions of concepts that interest us, found in the research of many specialists.

The distance from vertex u of a graph G to the vertex most distant from it is called the eccentricity of vertex u. The central vertex is the vertex of the graph for which the eccentricity takes the smallest value. The set of all central vertices is called the center of a graph G. We denote the number of central vertices of a graph G by $n(G)$. The diameter is the chain with maximum length. The sum of the distances of all vertices of the graph G from vertex u is called the transmission number of vertex u. The vertex of the graph with the smallest transmission number is called medial. The set of all medial vertices is called the median of a graph G. We denote the number of medial vertices of a graph G by $m(G)$. We can obtain the transmission numbers of graph vertices, for example, using Dijkstra or Lee algorithms.

It is known that the center of each tree consists of one vertex or two adjacent vertices. We have proven a similar result for the median of a tree: the median of each tree consists of one vertex or two adjacent vertices. From these results it follows that for every tree T the numbers $n(T)$ and $m(T)$ are equal to 1 or 2. We show that for these numbers all four logically possible relationships are possible. Moreover, one or two medial vertices may or may not be located on the diameter of the tree, and the distance between the central and medial vertices can be arbitrary.

Let $l(e)$ be a positive function of lengths defined on the set of edges of the graph G. The following concepts are defined accordingly: l-distance, l-eccentricity, l-diameter, l- transmission number, l-center, l-median. It is proven that similar results hold for l-centers and l-medians of such trees with weighted edges. Note that the results obtained can be generalized to other classes of both undirected and directed graphs with weighted vertices and edges.

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CONFORMAL DIMENSION OF THE BROWNIAN GRAPH

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joint work with
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Conformal dimension of a metric space X, denoted by $\dim_C X$, is the infimum of the Hausdorff dimensions among all its quasisymmetric defformations. If conformal dimension of X is equal to its Hausdorff dimension, X is said to be *minimal for conformal dimension*. In this talk we will give the first examples of minimal random fractals. Specifically, we will show that the graph of one dimensional Brownian motion is almost surely minimal for conformal dimension, which is equal to $3/2$. We also give other examples of sets that are minimal for conformal dimension. These include Bedford-McMullen self-affine carpets with uniform fibers as well as graphs of continuous functions of Hausdorff dimension d, for every $d \in [1, 2]$. The main technique in the proof is the construction of "rich families of minimal sets of conformal dimension one". The latter concept is quantified using Fuglede's modulus of measures.

CALCULATING THE MOORE-PENROSE INVERSE FOR TRIDIAGONAL SKEW-HERMITIAN MATRICES WITH ZERO DIAGONAL ELEMENTS

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Let us denote by A^* the conjugate transpose of a matrix A with complex elements. The matrix A is said to be a *skew-Hermitian* if $A^+ = -A$. Skew-Hermitian and, in particular, skew-symmetric matrices have many applications in various fields, such as computational mathematics, statistical analysis, signal processing and others [1].

As follows from the definition, the diagonal elements of a skew-Hermitian matrix are either purely imaginary numbers or zeros. In this work we discuss the case when all diagonal elements are zero. To find the Moore-Penrose inverse for tridiagonal matrices, we will use an approach developed in [2, 3] for skew-symmetric matrices. We are talking about generalized inversion, since tridiagonal skew-Hermitian matrices of odd order with zero diagonal elements are singular (it is easy to verify this). Recall that for a $m \times n$ matrix A the Moore-Penrose inverse A^+ is the unique $n \times m$ matrix that satisfies the following four conditions [4]:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (A^+A)^* = A^+A, \quad (AA^+)^* = AA^+.$$

If A is a square nonsingular matrix, then $A^+ = A^{-1}$. Note that the Moore-Penrose inverse of a skew-Hermitian matrix is also skew-Hermitian.

So we are considering a tridiagonal matrix

$$A = \begin{bmatrix} 0 & a_1 & & & \\ -\overline{a_1} & 0 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\overline{a_{n-2}} & 0 & a_{n-1} \\ & & & -\overline{a_{n-1}} & 0 \end{bmatrix}, \tag{1}$$

where $n \geq 3$. Note that throughout this report $\overline{z}$ stands for the complex conjugate of the complex number z. We assume that $a_i \neq 0$ for all $i = 1, 2, \ldots, n-1$. This requirement is not restrictive, since if some of the overdiagonal elements of A are equal to zero, the problem of computing the Moore-Penrose inverse is decomposed into several similar problems for matrices of lower order.

As regards the matrices of even order, i.e. $n = 2m$, then, according to above assumption about the overdiagonal elements, the matrix A is nonsingular and finding its inverse is not a difficult problem. Therefore, we will focus on the case when the matrix A from (1) is of odd order, i.e. $n = 2m+1$. Regardles of the values of overdiagonal elements, this matrix is singular. The computation of the Moore-Penrose inverse A^+ is based on a special representation of the matrix A.

Let us introduce bidiagonal matrix

$$B = \begin{bmatrix} -\overline{a_1} & a_2 & & & & \\ & -\overline{a_3} & a_4 & & & \\ & & \ddots & \ddots & & \\ & & & -\overline{a_{2m-1}} & a_{2m} \end{bmatrix} \tag{2}$$

of size $m \times m+1$. Next, we define the following matrices:

$$F = [f_{ij}]_{m \times 2m+1}, \quad f_{ij} = \begin{cases} 1, & \text{if } j = 2i \\ 0, & \text{if } j \neq 2i, \end{cases} \qquad i = 1, 2, \ldots, m$$

and

$$G = [g_{ij}]_{m+1 \times 2m+1}, \quad g_{ij} = \begin{cases} 1, & \text{if } j = 2i-1, \\ 0, & \text{if } j \neq 2i-1, \end{cases} \qquad i = 1, 2, \ldots, m+1.$$

Then the matrix A can be written as follows:

$$A = F^T BG - (F^T BG)^*.$$

In this work we show that for the matrix A^+ the following representation is valid:

$$A^+ = G^T B^+ F - (G^T B^+ F)^*.$$

So, the problem of finding the More-Penrose inverse for matrix A is reduced to a similar problem for matrix B given in (2). This goal is achieved in the proposed study.

Let us emphasize two main results of the work. First, we have obtained closed form expressions for the elements of the Moore-Penrose inverse of odd order tridiagonal skew-Hermitian matrices. Secondly, on the basis of the obtained formulas and relations, a numerical algorithm which is optimal in terms of computational costs was constructed.

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COMPOUND WIRETAP CHANNEL

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Investigation of communication over a **wiretap channel** is one of the problems of information - theoretic security [1]. The goal in designing communication systems in the presence of a wiretapper is to ensure that the message remains confidential between the transmitter and the intended receiver while minimizing the information available to the eavesdropper [2]. The advantage of information theoretic approach in security settings is the possibility of transmitting confidential messages without using an encryption key, resulting in lower complexity and savings in resources. The first information - theoretic task is to find the capacity of the model. The next task is the investigation of reliability function or E-capacity (rate-reliability function) suggested by E. Haroutunian [3]. In the model of wiretap channel the equivocation rate is added and the first aim is to investigate the capacity-equivocation region as well as the secrecy capacity, which was obtained in [4]. The analogy of E-capacity is the E-capacity-equivocation region, which is the closure of the set of all achievable rate-reliability-equivocation pairs. E-capacity-equivocation region and E-secrecy-capacity of wiretap channel are investigated by author in [5], where the outer and inner bounds are constructed. These results are, correspondingly, the generalizations of capacity-equivocation region and secrecy-capacity introduced and studied in [4], since the latter can be obtained from the corresponding constructed bounds as a particular case when E tends to 0.

The E-capacity is investigated for various multiterminal and varying channel models in [6] including **compound channel**, when the channel depends on parameter, which is invariable during transmission of one codeword, but can be changed arbitrarily for transmission of the next codeword. This model can be considered in four cases, when the current state of the channel is known or unknown at the encoder and at the decoder. The capacity of this channel was found by Wolfowitz [7], who has shown that the knowledge of the state at the decoder does not improve the asymptotic characteristics of the channel. So it is enough to study the channel in two cases. The upper and lower bounds of E-capacity in both cases are introduced in

[8].

Currently we investigate the compound wiretap channel model, which is the extension of the wiretap channel, when the channels to the legitimate receiver and to the wiretapper depends on the number of possible states. The capacity - equivocation region of the compound wiretap channel is unknown till now. The outer bounds of the E-capacity-equivocation region in both cases are constructed by author [9, 10]. Next step is the construction of inner bounds.

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ON THE CONSTRUCTION OF KERNELS OF TRANSFORMATION OPERATORS FOR DIRAC SYSTEM

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Let $\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are well-known Pauli matrices, and

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is known that the solution $y = \varphi(x, \lambda, \alpha)$ of the Cauchy problem

$$\begin{cases} \{\sigma_1 \frac{1}{i}\frac{d}{dx} + \sigma_2 p(x) + \sigma_3 q(x)\} y = \lambda y, \quad \lambda \in \mathbb{C} \\ y(0) = \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} \end{cases} \tag{1}$$

can be represented in the form

$$\varphi_0(x, \lambda, \alpha) = \begin{pmatrix} \sin(\lambda x + \alpha) \\ -\cos(\lambda x + \alpha) \end{pmatrix},$$

and

$$\varphi(x, \lambda, \alpha) = \varphi_0(x, \lambda, \alpha) + \int_0^x K(x,t)\varphi_0(t, \lambda, \alpha)dt = (E + K)\varphi_0.$$

Operator $E + K$ is called the transformation operator. Under different conditions on scalar functions p and q, this operator and its kernel $K(x,t)$ was investigated in different papers (see [1–6]).

Theorem. *Let $p, q \in L^1_{loc}(0, \infty)$. Then the kernel $K(x,t)$ and the kernel $H(x,t)$ of inverse operator $\varphi_0(x, \lambda) = \varphi(x, \lambda) + \int\limits_0^x H(x,t)\varphi_0(t, \lambda)dt$ can be represented in the form*

$$K(x,t) = a\sigma_1 + b\sigma_2 + c\sigma_3 + d \cdot E,$$
$$H(x,t) = \tilde{a}\sigma_1 + \tilde{b}\sigma_2 + \tilde{c}\sigma_3 + \tilde{d} \cdot E,$$

where the functions (of two variables (x,t)) a, b, c, d and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are represented by functions p and q.

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SOFT RIEMANN-HILBERT PROBLEMS AND PLANAR ORTHOGONAL POLYNOMIALS

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It has been known since the 2007 paper by Its and Takhtajan that the orthogonal polynomials with respect to an exponentially varying weight in the plane are characterized in terms of a 2×2 matrix dbar-problem. In the recent breakthrough by Hedenmalm and Wennman the asymptotics of these planar orthogonal polynomials was found (Acta Math, 2021). However, the connection with the 2×2 dbar-problem was left open for further investigation.

It turns out that there is a nice algorithm to find the asymptotics of the planar orthogonal polynomials in terms of the dbar-problem, which also has the benefit of supplying better error terms. This algorithm will be presented here. The work was published in CPAM 2024.

BIORTHOGONAL SYSTEMS, BASES AND INTERPOLATION BY MITTAG-LEFFLER FUNCTIONS IN SPACES $A^2_\omega(\mathbb{C})$

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The report presents some statements on biorthogonal systems, bases and interpolation in some of M. M. Djrbashian Hilbert space spaces $A^2_\omega(\mathbb{C})$ of entire functions with square integrable modulus over the entire complex plane. One of the systems consists of Mittag-Leffler functions.

THE LIPSCHITZ PROPERTIY OF A SET-VALUED MAPPING WITH WEAKLY CONVEX GRAPH

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In this paper, we examine set-valued mappings with weakly convex graphs and provide sufficient conditions for their Lipschitz properties.

The concept of weakly convex sets was introduced by Vial [3], and subsequent studies on their properties were conducted by Ivanov and Balashov [1, 2].

Definition 1 (see [2]). *Let $x_0, x_1 \in R^n$, $\|x_0 - x_1\| \le 2R$. The set*

$$D_R(x_0, x_1) = \bigcap_{a \in R^n : \{x_0, x_1\} \in B_R(a)} B_R(a)$$

is called a strongly convex segment of radius R and the set

$$D_R^o(x_0, x_1) = D_R(x_0, x_1) \setminus \{x_0, x_1\}$$

is called a strongly convex segment of radius R without extreme points.

Definition 2 (see [3]). *A subset A of R^n is called weakly convex with constant $R > 0$, if for $x_0, x_1 \in A$ such that $0 < \|x_0 - x_1\| < 2R$ the set $A \cap D_R^o(x_0, x_1)$ is nonempty.*

Let $a : R^n \to 2^{R^m}$ be a set-valued mapping. The graph of a defined as follows

$$graph(a) = \{(x, y) \in R^n \times R^m : y \in a(x)\}.$$

A set-valued mapping $a : R^n \to 2^{R^m}$ is called Lipschitz continuous if there exists a real constant $L \ge 0$ such that for all x_1 and $x_2 \in R^n$

$$H(a(x_1), a(x_2)) \le L\|x_1 - x_2\|,$$

where $H(a(x_1), a(x_2))$ is the Hausdorff distance between $a(x_1)$ and $a(x_2)$.

Let

$$dom(a) = \{x \in R^n : a(x) \ne \emptyset\}.$$

Theorem. *Let $a : R^n \to 2^{R^m}$ be a set-valued mapping with weakly convex (with constant R) and closed graph. Suppose $graph(a) \subset B_R(x_0, y_0)$.*

Then it is Lipschitz on a compact set $\Omega \subseteq int dom(a)$.

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GENERALIZED BALANCED FUNCTIONAL EQUATIONS ON n–ARY GROUPOIDS WITH QUASIUNITS

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Dedicated to Professor Yuri Movsisyan
colleague, coauthor and friend

Quasiunit of an n–ary groupoid $A : S^n \to S$ is an n–tuple $(a_1, \ldots, a_n)$ of elements from S such that for any i $(1 \le i \le n)$ translation $A_i(x) = A(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is a bijection.

This enables us (relative to some fairly mild assumptions) to generalize results of A. Krapež: On solving a system of balanced functional equations on quasigroups I–III, Publ. Inst. Math.(Beograd) (N.S.) 23(37) (1978), 25(39) (1979), 26(40) (1979) and solve any generalized balanced functional equation on n–ary **groupoids with quasiunits**. Solutions of such equations are similar to solutions of equations where we assume that unknown operations are quasigroups.

Compared to the case with just binary operations, we have a new case of **reducible** equations where there is at least one operation expressible as a term with operations of smaller arities. One of the simplest such cases is the equation $A(x, y, z) = E(x, F(y, z))$ which forces reduction of A :

$$A(x, y, z) = A_{12}(x, A_2^{-1} A_{23}(y, z)).$$

After all possible reductions we get an **irreducible** equation.

In irreducible equations we consider classes of principally isostrophic $(\sim)$ operations. For binary operations this reduces to diisotopies between them.

$\sim$–classes can be:

- Small (with only 2 operations)
- Big (with > 2 operations)

- Abelian (big and with operations isotopic to their duals).

In case of n–ary operations ($n > 2$), only the first case is possible, i.e. all $\sim$–classes of principal isostrophy between n–ary operations are small.

MINDFLAYER: EFFICIENT ASYNCHRONOUS PARALLEL SGD IN THE PRESENCE OF HETEROGENEOUS AND RANDOM WORKER COMPUTE TIMES

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We study the problem of minimizing the expectation of smooth non-convex functions with the help of several parallel workers whose role is to compute stochastic gradients. In particular, we focus on the challenging situation where the worker compute times are arbitrarily heterogeneous and random. In the simpler regime characterized by arbitrarily heterogeneous but deterministic compute times, Tyurin and Richtárik [1] recently proposed the first optimal asynchronous SGD method, called Rennala SGD, in terms of a novel complexity notion called time complexity. The starting point of our work is the observation that Rennala SGD can have bad and even arbitrarily bad performance in the presence of random compute times. To advance our understanding of stochastic optimization in this challenging regime, we propose a new asynchronous SGD method, for which we coin the name MindFlayer SGD, and perform theoretical time complexity analysis thereof. Our theory and empirical results demonstrate the superiority of MindFlayer SGD over existing baselines, including Rennala SGD.

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ON HYPERIDENTITIES OF ASSOCIATIVITY

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Let $(Q;\cdot)$ be a groupoid and $a \in Q$. Denote by L_a (R_a) the map of Q to Q such that $L_a(x) = ax$ ($R_a(x) = xa$) for all $x \in Q$. They are called left and right translations of the $(Q;\cdot)$. In detail we denote $L_{(\cdot),a}$ ($R_{(\cdot),a}$) for L_a (R_a).

A groupoid $(Q;\cdot)$ is said to be a division (cancellation) groupoid if L_a and R_a are surjective (injective) for every $a \in Q$. If the groupoid $(Q;\cdot)$ is a division (cancellation) groupoid then the operation $(\cdot)$ is called division (cancellation) operation, too. A quasigroup is a cancellation and division groupoid. A loop is a quasigroup $(Q;\cdot)$ with the unit e such that $ex = xe = x$ for any $x \in Q$.

A binary algebra $(Q;\Sigma)$ is called division (cancellation) algebra if $(Q;A)$ is a division (cancellation) groupoid for any operation $A \in \Sigma$. Division and cancellation algebra is called invertible algebra.

A groupoid $(Q;\cdot)$ is called left pre-cancellative if

$$ca = cb \rightarrow R_a = R_b,$$

where $a,b,c \in Q$. Right pre-cancellative groupoids are defined dually. A groupoid is called pre-cancellative if it is both left and right pre-cancellative. If groupoid $Q(A)$ is pre-cancellative then the operation A is also called pre-cancellative. A binary algebra $(Q;\Sigma)$ is called pre-cancellative if the groupoid $(Q;A)$ is pre-cancellative for any operation $A \in \Sigma$.

Every quasigroup is a pre-cancellative and division algebra.

Any division and pre-cancellative groupoid is called a pre-qusigroup. A binary algebra $(Q;\Sigma)$ is called pre-invertible algebra if $Q(A)$ is a pre-quasigroup for any operation $A \in \Sigma$.

In this talk we present the classification of hyperidentities of associativity in algebras with a pre-quasigroup operation.

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GENERALIZATION OF THE CONCEPT OF COVARIOGRAM FOR UNBOUNDED MEASURABLE SETS

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The concept of covariogram is extended from bounded convex bodies in $\mathbb{R}^d$ to the entire space $\mathbb{R}^d$ by obtaining integral representations for the distribution and probability density functions of the Euclidean distance between two d-dimensional Gaussian points that have correlated coordinates governed by a covariance matrix. When $d = 2$, a closed-form expression for the density function is obtained (see [1]). Precise bounds for the moments of the considered distance are found in terms of the extreme eigenvalues of the covariance matrix. We have defined the normal covariogram of $\mathbb{R}^d$ and established an analogous relationship to [2–4], including integral representations for the distribution and density functions of the Euclidean distance between two d-dimensional Gaussian points, characterized by correlated coordinates through a covariance matrix. Precise bounds for the moments of the considered distance in terms of the extreme eigenvalues of the covariance matrix are found (see [2]). When $d = 2$, an expression for the density function in terms of a modified Bessel function is obtained.

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ON DRINFELD MODULAR CURVES FOR SL(2)

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We study Drinfeld modular curves arising from the Hecke congruence subgroups of $\mathrm{SL}_2(\mathbb{F}_q[T])$. Using a combinatorial method of Gekeler and Nonnengardt, we obtain a genus formula for these curves. In cases when the genus is one, we compute the Weierstrass equation of the corresponding curve. This is joint work with Jesse Franklin and Sheng-Yang Kevin Ho.

ON THE ISSUE OF STUDYING AND ELIMINATING DEADLOCK SITUATIONS USING PETRI NETS

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The article explores some modeling issues for eliminating deadlock situations using Petri nets. An illustration of a model of the famous lunching philosophers problem using Petri nets is given. The Petri net model constructed by the author has the property of the idea of priority, which ensures synchronous regulation of the actions of lunching philosophers, which allows avoiding possible deadlocks in the system.

1. Introduction. Petri nets are modeling mechanisms that allow you to describe both the possible states of the system and its possible actions. The Petri net consists of four elements: P- a set of positions, T - a set of transitions, I - an input function, O- an output function. Positions describe the state, and transitions describe the actions taking place in the network. The network structure is a quartet of elements $C = (P, T.I, O)$, where P and T are finite sets of positions and transitions, and the input function I maps the transition to the set of input positions, $I(t_j)$ and the output function O maps the transition t to the set of output positions, $O(t_j)$, i.e. the transition can have both input and output positions, moreover, with repetitions. Let us give a formal definition of the classical Petri net.

Defenition. A Petri net is a pair $M(C, \mu)$, where $C = (P, T.I, O)$ is the structure of the net and μ is the state of the net. In the structure, C, P and T are finite sets of positions and transitions, $I : T \to p^{\infty}$, $O : T \to P^{\infty}$ are input and output functions, respectively, where P^{∞} are all possible (with repeating elements) sets $P, \mu : P \to N_0$ is a state function, where $N_0 = \{0, 1, \dots\}$ is the set of all positive integers. Permissible transitions of the Petri net are determined in a known way, and the transition of the network from state μ to state μ is the set of reachable states [1-5].

2. The Dining Philosophers Problem. On the table is a bowl of rice, five plates and five chopsticks. If a philosopher is hungry, he enters the dining room, takes a free seat at the table, takes two (mandatory!) chopsticks and puts them on a plate (Fig. 1). Having satisfied his hunger, the philosopher returns the chopsticks to the table and leaves the dining room. In the event that all five philosophers come to the dining room at the same time, take their places at the table, take a stick each, the system will be blocked, because. none of the philosophers will be able to start eating (Fig. It is required to organize the system in such a way that five philosophers cannot be at the table at the same time. If this happens, the system will be blocked.

Figure 1: Five Dining Philosophers Problem.

3. Synchronous regulation of the actions of dining philosophers using a Petri net Let's build a model for solving this problem using a Petri net. Figure 2 shows the Petri net, which shows the main actions of one philosopher, but it should be borne in mind that other philosophers act in the same way, and when visiting the dining room, they can simultaneously try to take one and the same stick. Therefore, a mutex is needed, which will organize access to the process of selecting sticks in order of priority. Also, each stick is assigned a mutex, which organizes access to this stick in turn [1].

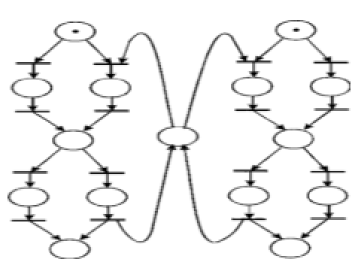

Figure 2: Solving the problem of dining philosophers using the methodology of K. Petri.

From the resulting model (Fig. 3) it can be seen that all actions can occur sequentially, but following the above condition.

The position P_n and transition T_n ensure that all actions are performed synchronously while 5 tokens are collected in the position P_n, i.e. until 5 actions are performed it cannot be performed T_n, and the actions each receive one initial marking $T_i \quad (1 \leq i \leq 5)$.

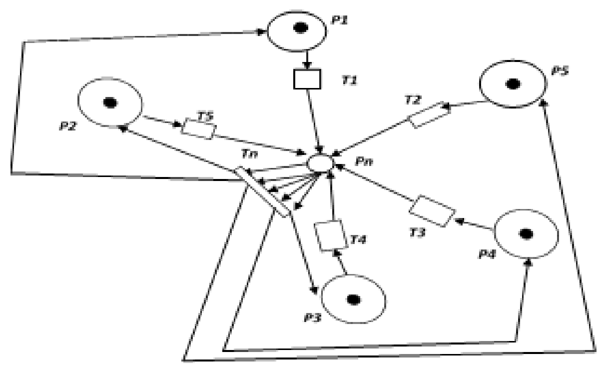

Figure 3: Synchronous regulation of the actions of dining philosophers using a Petri net.

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ON CYCLIC INTERVAL COLORINGS OF GRAPHS

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A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \rightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. A proper edge-coloring of a graph G with colors $1, 2, \ldots, t$ is called a *cyclic interval t-coloring* if for each vertex v of G the edges incident to v are colored by consecutive colors, under the condition that color 1 is considered as consecutive to color t. A graph G is called *cyclically interval colorable* if it has a cyclic interval t-coloring for some positive integer t. Let $\mathcal{N}_c$ be the set of all cyclically interval colorable graphs. For a graph $G \in \mathcal{N}_c$, we denote by $w_c(G)$ and $W_c(G)$ the minimum and maximum number of colors in a cyclic interval coloring of a graph G, respectively. A graph G is *convex-round* if its vertices can be circularly enumerated such that the open neighbourhood of every vertex is an interval in the enumeration. In this talk, we show that if G is a convex-round graph of order n, then $G \in \mathcal{N}_c$ and $w_c(G) \leq n$. In particular, this result implies that all complete multipartite graphs, circular cliques and circular doubly convex balanced bipartite graphs are cyclically interval colorable. We also show that if G is an outerplanar graph with $\Delta(G) \geq 3$ and $G \in \mathcal{N}_c$, then $W_c(G) \leq |V(G)| + \Delta(G) - 3$, where $\Delta(G)$ is the maximum degree of G. Moreover, all these bounds are sharp.

TOWARDS ADEQUATE MODELS OF ORIGINATION AND DEVELOPMENT OF COGNIZING

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1. Humans cognize allover including themselves.
   Interpreting developmental psychology by Piage (Flavell1962), cognizing is *learning and organizing mental systems* (mss) *to promote human utilities*.
   1.1 Piaget successfully tracked development of cognizing from newborns to the highest human one, but its origin still challenging the researchers.
   1.2 In our studying we construct models of cognizers, argue their adequacy and provide premises of their origination in nature (Pogossian 2020-24). Let's outline this findings.
2. The models of *generalized cognizers* are defined as realities with energizers and certain utilities that throughout their lifetime regularly and unlimitedly learn and organize certain constructions, *mentals*, to promote their utilities.
   2.1. The definition of mentals (generally exempted from cellular and computer dependency) is incremental and is based on those of doers, sensors, classifiers, relationships, attributes, imprints, identifiers, nominals, doins, systems over nominals and others.
3. In **justification of adequacy of cognizers** tending to be carried out by analogy with justification of algorithms as adequate models of computability by Church, we argue that our models
   – are completely explainable
   – preserve the majority of known statements and algorithms of cognizing including
     = inductive learning algorithms, particularly in the Neuron Nets (NN) mode,

  - = Personalized Planning/Integrative Testing algorithms elaborating strategies in target situations dependent on the learned classifiers, thus, elaborating "if then" relationships - the base for formation algorithms, say, by A. Markov or E. Post,
  - = algorithms of acquisition of strategy meanings by experts and those from the texts (Pogossian 2020, Grigoryan 2021) conceptually close to (Langley, Shrobe, Katz, 2020),
- – provide expert like explanations/interpretations of mentals
- – can be based on any classifiers, say on NN, thus, consisting functional and connectivity models of cognizing
- – successfully approximate expert solutions of security, competition and dialogue HU* caseproblems (Pogossian 2020)
- – are supportive to revelation of origination of cognizing.

4. **Questioning origination of cognizing** (Pogossian 2020-24) should, first of all, turn to the origination of cognizing of living realities, i.e., cellular, and, as a minimum, of the simplest cellular, uncials. By one of the prevalent hypotheses, *abiogenesis*, uncials, were originated by chance from chemical compounds already existed in nature. Unfortunately, despite of ongoing intensive research efforts, abiogenesis holds more difficulties and hopes than advances Paul Dirac (2007), (Dembski 2007), (Irreducible complexity).

   4.1. While studies on abiogenesis continue, new ideas and hypotheses on the origin of uncials emerge attempting to exempt from the difficulties of abiogenesis.

   By the hypothesis on *origin-able cognizing in nature* (oacin), arisen in constructive modeling of cognizing, cellulars were designed by a type of cognizers of the Universe which

   - – were earlier originated in nature as elementary recurrent classifiers, then
   - – evolving had attained the power of cognizing comparable, at least, to the highest human one, followed by
   - – designing cellular, analogous to human design of robots nowadays.

   4.2. Viability of oacin hypothesis is strengthened by assertions that

   - – constructions, mentals, adequately model mss

- mss and means of their construction can be composed of elementary “atoms”, *recurrent 1-/2-place classifier*
- a type of constructive cognizers, *octaves*, exempted, generally, from computer dependencies and capable of enhancing the power of cognizing throw learning mentals, but so far limited in that, can adequately model cognitive development of newborns by Piaget
- octaves, and assumingly their *roots*, can be reduced to some alphabet of uniform units, i.e., inevitable constituents of cognizers
- studying the origination of octaves/roots can be based on studying the origination of their constituents
- functional definition of constituents of octaves/roots softens the requirements to their implementations.

5. **Upcoming research** in the origination and development of cognizers reduces, particularly, to origination of the dynamicity of doers, energizers and their ability to develop to octaves and other unavoidable constituents including doers of the types of 1/2 place symbolic and non-symbolic recurrent classifiers comprising strategy/algorithms case and rule based nets, compartments and reproducers of cognizers.
   - 5.1. Note that while some of the above questionings are analogous with ones of abiogenesis, an advantage of oacin is in its functional questionings, what can essentially extend the variety of solutions.
   - 5.2. Note also that these studies along with enriching applications of current cognizing models, if successful, will support to shed light on the fundamental question of the origin of cellular, and thus, of humans.

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BARYCENTRIC ALGEBRAS AND BARYCENTRIC COORDINATES

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Real affine spaces are (abstract) algebras with non-associative binary operations indexed by real numbers. Convex subsets are subalgebras under the operations indexed by real numbers taken from the open unit interval. The algebras defining convex sets generate the variety of *barycentric algebras*. Convex polytopes considered as barycentric algebras are generated by their vertices. In particular, simplices are free barycentric algebras over their vertex sets. Each element of a simplex is presented as a convex combination of its vertices, with barycentric coordinates defined in a unique way. Each general convex polytope is a homomorphic image of a simplex. Hence each of its elements can also be presented by convex combinations, but not necessarily in any unique way. Thus, the following problem is important in many applications of polytopes:

> Given the set of vertices of a convex polytope, determine algorithms for the barycentric coordinates of each point of the polytope.

There exist several methods of solving the problem for specific convex polytopes, leading to different systems of barycentric coordinates.

We introduce the general concept of a *coordinate system* on a polytope, and show that the coordinate systems on a polytope themselves form a convex set. We present new coordinate systems that are based on decompositions of a convex polytope into unions of simplices. For the case of polygons (2-dimensional polytopes), these systems exhibit interesting combinatorial properties that relate to the parsing trees of non-associative products and coproducts.

This is joint work with Jonathan D.H. Smith and Anna Zamojska-Dzienio.

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ON THE MULTIPLICITY OF EIGENVALUES OF ONE SYMMETRIC MATRIX OF THE 5TH ORDER

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We consider the following symmetric matrix of constant coefficients (see, for example, [1]-[3]) of order 5

$$A = \{a_{ij}\}, \quad a_{ij} = a_{ji}, \qquad i, j = 1, 2, 3, 4, 5.$$

It is assumed that the matrix elements are equal to a or b. The purpose of this article is to use the method of eigenvalues to consider and generalize cases in which the characteristic equation of matrix A will have a fourfold, threefold or twofold root(s). Obviously, this will make it possible to simulate such simmetric matrices whose characteristic equation $det(A - \lambda E)$, where E- is unit matrix 5th order, will have a root(s) of a given multiplicity.

Theorem 1. *If one of the conditions*

$$a_{ik} = a_{ij} = a_{kj} = a_{mn}$$

is met in the matrix A, while the remaining elements are equal $b \neq a$, then the characteristic equation has a threefold root

$$\lambda_1 = \lambda_2 = \lambda_3 = -a, \qquad \lambda_{4,5} = \frac{3a \pm \sqrt{a^2 + 24b^2}}{2}.$$

Here and everywhere in what follows it is assumed that the indices i, j, k, m, n are different.

Theorem 2. *If one of the conditions*

$$a_{ij} = a_{ik} = a_{im} = a_{in}$$

is met in the matrix A and the remaining elements are equal $b \neq a$, then the characteristic equation has a threefold root:

$$\lambda_1 = \lambda_2 = \lambda_3 = -b \qquad \lambda_{4,5} = \frac{3b \pm \sqrt{16a^2 + 9b^2}}{2}.$$

Theorem 3. *If one of the conditions*

$$a_{ij} = a_{jk} = a_{km} = a_{mn} = a_{ni}$$

is satisfied in the matrix A and the remaining elements are equal $b \neq a$, then the characteristic equation has two double roots:

$$\lambda_1 = \lambda_2 = -\frac{a+b}{2} + \frac{1}{2}\sqrt{5}(b-a), \qquad \lambda_3 = \lambda_4 = \frac{a+b}{2} - \frac{1}{2}\sqrt{5}(b-a).$$

Theorem 4. *If one of the conditions*

$$a_{ij} = a_{jk} = a_{km} = a_{mi},$$

is satisfied in the matrix A and the remaining elements are equal $b \neq a$, then the characteristic equation has exactly one double root:

$$\lambda_1 = \lambda_2 = -b.$$

Theorem 5. *If one of the conditions*

$$a_{ik} = a_{km} = a_{mi} = a$$

is satisfied in the matrix A and th the remaining elements in the matrix are equal $b \neq a$, then the characteristic equation has one double root

$$\lambda_1 = \lambda_2 = -a$$

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SUMMATION FORMULAE FOR SERIES OVER THE ZEROS OF THE ASSOCIATED LEGENDRE FUNCTIONS WITH PHYSICAL APPLICATIONS

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The solutions of the wave equation in background of a constant negative curvature space are expressed in terms of the associated Legendre functions. In problems with spherical boundary the eigenmodes of the radial quantum number inside the sphere are expressed in terms of the zeros of the associated Legendre function of the first kind with respect to the degree. By using the generalized Abel-Plana formula we present a summation formula for series over those zeros. The formula is applied for the evaluation of the Wightman function for a scalar field obeying the Robin boundary condition on the sphere. That allows to explicitly separate the Wightman function in the geometry without the sphere and to present the sphere induced contribution in terms of strongly convergent integral. In this way the renormalization of the expectation values of physical observables is reduced to the one in the boundary-free geometry.

UNORTHODOX ALGEBRAS AND UNORTHODOX LOGICS

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- Let me start with the question: What is $F \to T =$? (or, $0 \to 1 =$?)
- Of course, the usual answer is: $F \to T = T$. Or, $0 \to 1 = 1$.
- Question: Can it be different from 1?
- Let us look at a very simple algebra:

$\bar{2}$:

$\to$	0	1
0	1	0
1	0	1

- Observe: $0 \to 1 = 0$. So, we will call it an "**anti-classical**" algebra. YET,
- It has an interesting logic.
- The logic corresponding to the variety $\mathbb{V}(\bar{2})$ is a **connexive logic**: Aristotle's Theses: $\vdash \neg(\neg\alpha \to \alpha)$, and $\vdash \neg(\alpha \to \neg\alpha)$; Boethius Theses: $\vdash (\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$ and $(\alpha \to \neg\beta) \to \neg(\alpha \to \beta)$.
- $\mathbb{V}(\bar{2})$ and $\mathbb{V}(2)(= BA)$ are **term-equivalent**!
- Hence, the classical logic is also a "connexive logic".
- Now, in view of these observations, the following question arises:
- **Question**: Are there "interesting" logics in which $0 \to 1$ is different from both 0 and 1?
- **The rest of my talk will address this question.**
- So, let us look at the following FIVE algebras in the language: $L = \langle \vee, \wedge, \to, ', 0, 1 \rangle$.

C^* AND CLIFFORD ALGEBRAS IN THE SOLID STATE PHYSICS

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After the discovery of quantum Hall effect and its topological explanation the mathematical methods based on the theory of C^*-algebras and their K-theory enter firmly into the arsenal of solid state physics.

A key role in the theory of solid states is played by their symmetry groups. It was Kitaev who has pointed out the relation between the symmetries of solid bodies and Clifford algebras.

In our talk we pay main attention to the class of solid bodies called the topological insulators. They are characterized by having a broad energy gap stable under small deformations. The algebras of observables of such solid bodies belong to the class of graded C^*-algebras for which there is a variant of K-theory proposed by Van Daele. It makes possible to define the topological invariants of insulators in K-theory terms.

LATTICE CHARACTERIZATION OF CLASSES OF GROUPS

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We present an overview of our recent results related to the lattice characterization of groups. We use the weak congruence lattice of a group G Wcon(G), which is a lattice extension of the subgroup lattice Sub(G): it consists of all normal subgroups of all subgroups of G, represented by the corresponding congruences on subgroups. For all these relations the domain is the whole group G. In this lattice, the normality relation among subgroups has an equivalent lattice description. Further, the weak congruence lattice of every subgroup is the principal ideal generated by the square of the subgroup and the weak congruence lattice of the quotient subgroup G/H is the principal filter generated by H^2. Using these and other similar features of the lattice Wcon(G), we were able to characterize numerous known classes of groups by lattice properties. Namely, we give necessary and sufficient conditions which should be fulfilled by the weak congruence lattice, under which a group is Dedekind, Hamiltonian, abelian, solvable, metabelian, perfect, supersolvable, nilpotent and others. In addition, we describe algebraic properties of several new classes of groups characterized by properties of their weak congruence lattices.

This is a joint research with A. Tepavčević, J. Jovanović and M. Grulović.

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ON DIFFERENTIAL ALGEBRAS WITH COMPOSITION

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The composition of functions often plays an important role in many domains of classical analysis.[1] However, its algebraic modeling lags behind the other analytic constructions like differentiation or considering the (normed) algebras of functions on the compacts. The present talk suggests some concepts that hopefully provide a basis for the corresponding algebraization; it develops the material of [2] with the emphasis of application.

For the time being even the appropriate <u>notations</u> are not yet developed. E.g., the *discrete dynamics* studies the behavior of the trajectories

$$x \mapsto f(x) \mapsto f\Big(f(x)\Big) \mapsto f\Big(f\Big(f(x)\Big)\Big) \mapsto \ldots,$$

where $x \in X$ is an element of the *phase space* X, which is a set with some structures and $f : X \to X$ is an endomorphism respecting these structures. The notational issue concerns the mapping $\underbrace{f \circ \cdots \circ f}_{n}$ for which I strongly suggest $f^{n\circ}$. In the current literature (see, e.g. [1]) the notation f^n is mostly used instead. It is disastrous: in the (highly non-trivial) case of 1-parametric family of functions $f : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto x^2 + c$ with $c \in \mathbb{R}$ the expression f^2 can mean either $f \cdot f = x^4 + 2cx^2 + c^2$ or $f \circ f = (x^2 + c)^2 + c = x^4 + 2cx^2 + c^2 + c$. In our approach we distinguish strictly the *algebraic* and the *compositional* powers and, more generally, operations.

We axiomatize the <u>three</u> binary and one unary operations carried by univariate polynomial rings.

Definition. Let $\mathbb{k}$ be a field. A $\mathbb{k}$-algebra $\mathbf{A}$ is called a *differential* $\mathbb{k}$*-algebra with composition* if it is endowed with a $\mathbb{k}$-differentiation $a \mapsto a'$ and with such an additional binary operation $\circ$ (called *composition*) that for all $a, b, c \in \mathbf{A}$

[1] We don't consider the category theory where the composition of morphisms belongs to the most basic structures.

1. $(a \circ b)' = (a' \circ b)b'$;
2. $(a + b) \circ c = a \circ c + b \circ c$;
3. $(ab) \circ c = (a \circ c)(b \circ c)$.

It is also natural to assume the existence of the bilateral *compositional unit*, i.e. such an element $z \in \mathbf{A}$ that $z \circ a = a \circ z = a$ for every $a \in A$.

Besides the polynomial algebras $\Bbbk[z]$ the algebra of *entire functions* $\mathcal{O}[\mathbb{C}]$ satisfies the above axioms.

Some applications to the polynomial dynamics will be demonstrated. In particular, for the polynomial $f = z^2 + c \in \Bbbk(c)[z]$ the polynomial

$$\frac{f^{2^n \circ} - z}{f^{2^{n-1} \circ} - z}$$

of degree $2^{2^n} - 2^{2^{n-1}}$, defining the 2^n-cycles of the mapping $z \to z^2 + c$, will be considered. One of the particular result, concerning the 4-cycles, is the following one:

Theorem. *The equation in z*

$$\frac{\left(\left((z^2+c)^2+c\right)^2+c\right)^2+c-z}{(z^2+c)^2+c-z} \equiv z^{12} + 6cz^{10} + \cdots + 2c^2 + 1 = 0$$

is solvable over $\Bbbk(c)$ *in radicals.*

However, the polynomial formalism turns out to be insufficient for the other classical problems. E.g., the *Schwarzian derivative* $f \mapsto \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ behaves nicely with respect to composition, but this behavior can not be formulated straightforwardly in the above terms. The needed categorical, sheaf-theoretic and universal-algebraic approaches to the extensions of the concept of differential algebras with composition will be discussed.

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STRANGE NON-LOCAL OPERATOR HOMOGENIZING THE POISSON EQUATION WITH DYNAMICAL UNILATERAL BOUNDARY CONDITIONS: CRITICAL CASE AND ARBITRARY SHAPE OF PARTICLES

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We study the homogenization of a nonlinear problem given by the Poisson equation, in a domain with arbitrarily shaped perforations (or particles) and with a dynamic unilateral boundary condition (of Signorini type), with a large coefficient, on the boundary of these perforations (or particles). The problem arises, for instance, in the study of chemical reactions of zero order. The consideration of a possible asymmetry in the perforations (or particles) is fundamental in order to consider some applications in nanotechnology were symmetry conditions are too restrictive. As a matter of facts, it is important also the consideration of perforations (or particles) constituted by small different parts and then with several connected components.We are specially concerned with the so called critical case in which the relation between the coefficient in the boundary condition, the period of the basic structure, and the size of the holes (or particles) leads to the appearance of an unexpected new term in the effective homogenized equation. Due to the dynamic nature of the boundary condition this “strange term” becomes now a non-local in time and non-linear operator. We prove the convergence theorem and find several properties of the “strange operator” showing that there is a kind of regularization through the homogenization process.

SKOLEMIZED QUANTUM QUASIGROUPS

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A major step forward in the development of Hopf algebra theory is its extension to the genuine non(co)associative case, beyond mere relaxation of the strictness of the monoidal category in which a Hopf algebra lives. The prototype is the extension of group theory to quasigroups. From the algebraic point of view, the combinatorial nature of quasigroup cancellation (in the finite case, a Latin square multiplication table) is not well behaved. For good behavior, quasigroups Skolemize to *equational quasigroups*, where the existence and uniqueness requirements of cancellativity are encoded in universally quantified identities satisfied by right and left division operations that augment the multiplication in the algebraic structure.

Initial attempts to incorporate genuine nonassociativity in Hopf algebras were made by the *Hopf quasigroups* of Majid et al. [3, 4], and by the *Hopf algebras with triality* of Benkhart et al. [1, 2, 5]. These structures lack the self-duality that is an essential feature of Hopf algebras. Around a decade ago, *quantum quasigroups* emerged as the truly self-dual non(co)associative extension of Hopf algebras [6]. So far, they have been defined merely as *bimagmas* (Q, ∇, Δ), equipped with mutually homomorphic multiplication $\nabla\colon Q \otimes Q \to Q$ and comultiplication $\Delta\colon Q \to Q \otimes Q$, satisfying invertibility of the *left composite* morphism

$$\mathsf{G}\colon Q \otimes Q \xrightarrow{\Delta \otimes 1_Q} Q \otimes Q \otimes Q \xrightarrow{1_Q \otimes \nabla} Q \otimes Q$$

and *right composite* morphism

$$\mathsf{Ə}\colon Q \otimes Q \xrightarrow{1_Q \otimes \Delta} Q \otimes Q \otimes Q \xrightarrow{\nabla \otimes 1_Q} Q \otimes Q$$

in the underlying monoidal category. Now, *equational quantum quasigroups* introduce auxiliary quantum quasigroups that offer explicit inverses for the composites. They come in three flavors: *quantum* S-, T-, and U-*quasigroups* Figure 1. All Hopf algebras are quantum S-quasigroups. If a Hopf algebra forms a quantum T-quasigroup, then it has certain grouplike properties that are not yet fully understood.

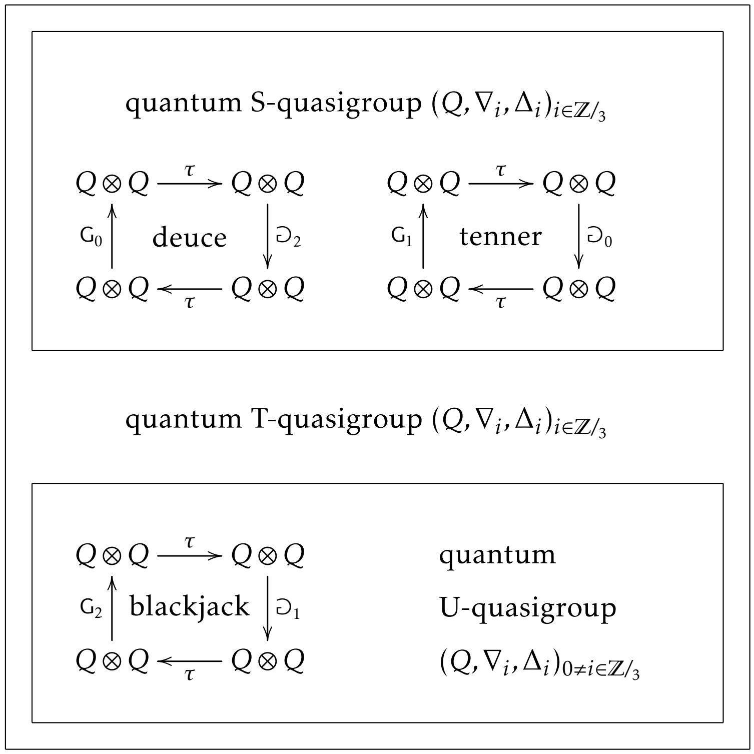

Figure 1: The equational quantum quasigroup definitions require commuting of the indicated named diagrams on a pair $(Q, \nabla_i, \Delta_i)_{0 \neq i \in \mathbb{Z}/_3}$ or triple $(Q, \nabla_i, \Delta_i)_{i \in \mathbb{Z}/_3}$ of bimagmas in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with swap τ.

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ABOUT QUADRATIC QUASIGROUP IDENTITIES

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We consider the class of quadratic functional equations over a *quasigroup environment* (briefly, *qen*), i.e. over any set $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2$ of functions defined on the same carrier set, where Φ_0 is a set of nullary operations, Φ_1 a group of bijections, Φ_2 is a set of binary quasigroup operations and the set Φ is closed under compositions of binary and unary or nullary operations as well as unary and nullary operations from Φ. A term is called *repetition-free*, if each individual variable appears at most once.

An equation is called: *balanced*, if both sides of the equation are repetition-free and they have the same set of individual variables; *quadratic*, if each individual variable appears twice or never; *cancellable*, if it has a subterm containing all appearances of an individual variable and none appearances of another; *parastrophically cancellable*, if it is parastrophically equivalent to a cancellable functional equation; *reducible over a qen*, if it is equivalent to a system of functional equations over the qen every of which has less number of different individual variables than the given one. Other definitions and results one can find in [1–3].

Theorem 1. *Each cancellable quadratic functional equations over a quasigroup environment is reducible over this environment.*

Theorem 2. *Each functional equation of mediality is parastrophically noncancelable, but reducible.*

Theorem 3. *Consider a quadratic functional equations in n individual variables over a quasigroup environment Φ. When $n > 4$, then the equation is parastrophically cancellable; when $n \leqslant 4$, then it is parastrophically cancellable or parastrophically equivalent to: unipotency if $n = 1$; commutativity if $n = 2$; associativity if $n = 3$; mediality if $n = 4$. The functional equations of unipotency, commutativity, associativity and mediality are parastrophically noncancellable.*

Corollary 1. *Any quadratic functional equations in the variety of all unipotent loops defines the variety of all unipotent loops, or the variety of all commutative unipotent loops, or the variety of all groups of the exponent two.*

Corollary 2. *Any quadratic functional equation on a qen is equivalent to a system of identities of commutativity, mediality and associativity on this qen.*

A quadratic identity is called *gemini*, if it is true in any Steiner loop.

Corollary 3. *Any quadratic quasigroup identity is gemini or equivalent to associativity in the variety of Steiner loops.*

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ABOUT PROLONGATIONS OF LATIN CUBES

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The first steps in this topic were made by Fedir Sokhatsky and his student Diana Kirku. In this work, we improve and explore more deeply the prolongations of Latin cubes. In particular, the cubes are given in the coordinate system $Oxyz$. We mark each cell (x,y,z,u), where (x,y,z) are coordinates of the cell, u is the element located in this cell. It will be also called the fourth coordinate of the cell. So, the Latin cube is denoted by

$$C := \{(x,y,z,u) \mid 0 \leqslant x,y,z,u < m\}. \tag{1}$$

The row plane, column plane, and string plane of the cube defined by an element a is denoted by the equations $x = a$, $y = a$, $z = a$ respectively. And the row $L_{1,a,b}$, column $L_{2,a,b}$ and string $L_{3,a,b}$ is denoted by a pair of the planes, i.e., by their intersection:

$$y = a, z = b; \qquad x = a, z = b; \qquad x = a, y = b.$$

If any cell of a set $T \subset C$ uniquely defined by: 1) an arbitrary pair of its coordinates, then T is called a *two-dimensional transversal*; 2) an arbitrary coordinate, then T is called the *one-dimensional transversal* of this Latin cube. The two-dimensional transversal of a Latin cube is a Latin square, and its arbitrary transversal is the one-dimensional transversal of the cube.

An algorithm for prolongations of Latin cubes. The goal of this algorithm is to construct a Latin cube by adding one new element to a given Latin cube.

Let (1) be a Latin cube of order m, in which there is a two-dimensional transversal τ and a one-dimensional transversal $\theta \subset \tau$.

Step 1. We add three new planes with empty cells to C, namely $x = m$, $y = m$, $z = m$.

Step 2. We transfer all elements from the cells of the two-dimensional transversal τ, except for the cells of the one-dimensional transversal θ, to the added planes. Namely, let $(a,b,c,d) \in \tau \setminus \theta$. We move d into three cells (m,b,c), (a,m,c) and (a,b,m). As a result, we get cells (m,b,c,d), (a,m,c,d) and (a,b,m,d), respectively, in the planes $x = m$, $y = m$, $z = m$. The cell (a,b,c) becomes empty.

Step 3. We copy the element d from each cell (a,b,c,d) of the one-dimensional transversal θ to the row $y = z = m$, column $x = z = m$, and string $x = y = m$. As a result, we get the cells (a,m,m,d), (m,b,m,d) and (m,m,c,d). The cell (a,b,c,d) is not empty.

Step 4. We put a new element m in the empty cells.

Step 5. The prolongation is completed.

Example. Let's prolongate the Latin cube of order 5 to the Latin cube of the order 6 (Fig. 1, the full Latin cube of the order 5 is given by $f(x,y,z) = 4x + 2y + 2z \pmod 5$).

In this figure two-dimensional transversal τ (the light blue circles) and the one-dimensional transversal θ (the blue circles) are selected in it. The new element is 5.

Step 1. Add new planes $x = 5$, $y = 5$, $z = 5$ to C.

Step 2. Let's transfer the elements of the two-dimensional transversal, except for the cells of the one-dimensional transversal ($(a,b,c,d) \in \tau \setminus \theta$), to the added planes in the cells $(5,b,c)$, $(a,5,c)$ and $(a,b,5)$. As a result, we will get the cells $(5,b,c,d)$, $(a,5,c,d)$ and $(a,b,5,d)$, respectively, in the planes $x = 5$, $y = 5$, $z = 5$. The cell (a,b,c) becomes empty.

Step 3. We copy the fourth coordinates from the cells of the one-dimensional transversal

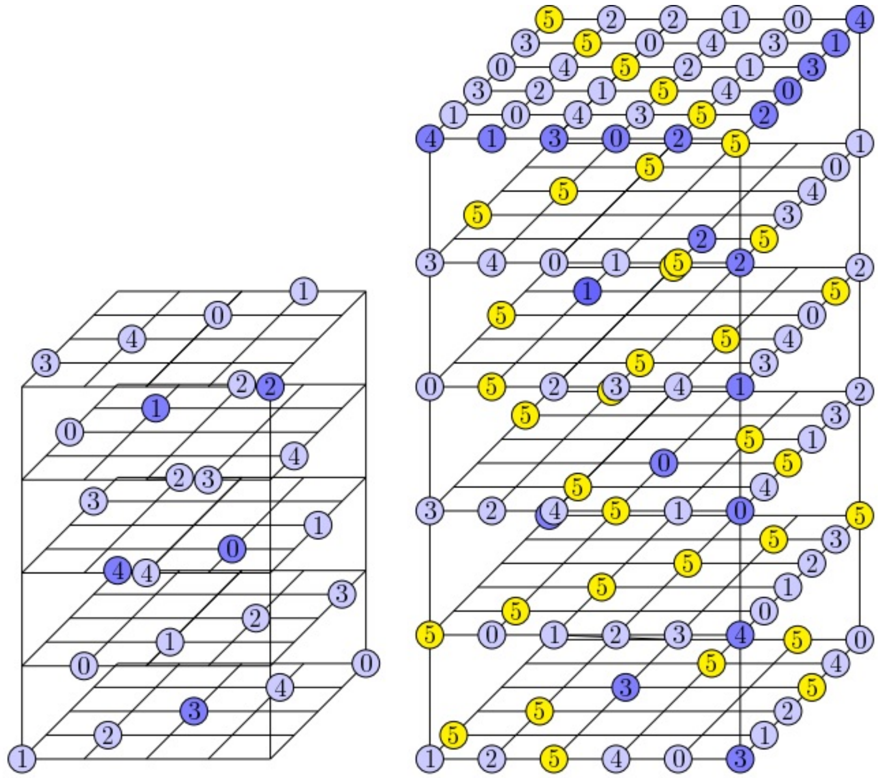
$$(2,0,2,3), \quad (0,1,0,4), \quad (3,2,3,0), \quad (1,3,1,1), \quad (4,4,4,2)$$

into the cells with coordinates $(a,5,5)$, $(5,a,5)$ and $(5,5,a)$, where $a \in \mathbb{Z}_5$. As a result, we get

row y=z=5:	$(0,5,5,4)$	$(1,5,5,1)$	$(2,5,5,3)$	$(3,5,5,0)$	$(4,5,5,2)$
column x=z=5:	$(5,0,5,3)$	$(5,1,5,4)$	$(5,2,5,0)$	$(5,3,5,1)$	$(5,4,5,2)$
string x=y=5:	$(5,5,0,4)$	$(5,5,1,1)$	$(5,5,2,3)$	$(5,5,3,0)$	$(5,5,4,2)$

Step 4. We put the new element 5 into the empty cells (the yellow circles).

Step 5. The prolongation is completed: in the second figure, a cube of the order 6 was formed.



THEORY OF LIGHT PROPAGATION IN 1D ARBITRARY INHOMOGENEOUS ISOTROPIC MEDIA

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The problem of electromagnetic wave propagation in a one-dimensional isotropic medium with an arbitrary or random distribution of the refractive index has practical applications in many fields. It can be solved using many methods, such as transfer matrix and scattering matrix methods [1], Green's function method [3], invariant embedding method [4], phase function method [5], and others [6–11]. Probably the most popular method is the Transfer Matrix Method with its variations. The mathematical basis of the method is the linearity of the electromagnetic field equations, which allows the linear matrix algebra apparatus to be applied within the framework of this method [12]. Each of these methods has its advantages and disadvantages. The choice of one or another method is usually determined by the problem formulation, namely, which aspect of the problem is of the greatest interest, the computational complexity, and the possibility of obtaining analytical expressions. A new method for determining the reflection and transmission amplitudes of an arbitrarily polarized plane wave, incident on the 1D inhomogeneous layer with an arbitrary dependence of dielectric permittivity $\varepsilon(z)$ and magnetic permeability $\mu(z)$ was suggested in paper [13] and then generalized in [14, 15]. This method is based on the transfer matrix method and reduces this problem to the Cauchy problem for a system of two first-order linear differential equations with given initial conditions. According to this method, the amplitudes of transmission $t^{p,s} = \frac{E_t^{p,s}}{E_i^{p,s}}$ and reflection $r^{p,s} = \frac{E_r^{p,s}}{E_i^{p,s}}$ of p- and s- polarized waves are expressed in terms of real functions $F_k^{s,p}(z)$

and $Q_k^{s,p}(z)$ by means of formulas [15]

$$t^{s,p} = \frac{2\exp(-ikL)}{Q_k^{s,p}(L) + F_k^{s,p}(L)}, \quad r^{s,p} = \frac{Q_{-k}^{s,p}(L) - F_{-k}^{s,p}(L)}{Q_k^{s,p}(L) + F_k^{s,p}(L)},$$

and these functions are solutions to the following differential equations:

$$\frac{dF_k^{s,p}}{dz} = -iA^{s,p}Q_k^{s,p}, \quad \frac{dQ_k^{s,p}}{dz} = -iB^{s,p}F_k^{s,p}, \tag{1}$$

with boundary conditions $F_k^{s,p}\Big|_{z=0} = 1$, $Q_k^{s,p}\Big|_{z=0} = 1$.

Here $E_{i,r,t}^{p}$ and $E_{i,r,t}^{s}$ are the p- and s- components of amplitudes for the incident, reflected and transmitted waves, correspondingly, $k = \left(\frac{\omega}{c}\right)\sqrt{\varepsilon(z)\,\mu(z)}\cos\beta(z)$, $A^s = \frac{\omega}{c}\sqrt{\frac{\varepsilon_s}{\mu_s}}\frac{1}{\mu(z)\cos\alpha}\left(\mu(z)\varepsilon(z) - \mu_s\varepsilon_s\sin^2\alpha\right)$, $B^s = \frac{\omega}{c}\mu(z)\sqrt{\frac{\mu_s}{\varepsilon_s}}\cos\alpha$, $A^p = \frac{\omega}{c}\varepsilon(z)\sqrt{\frac{\varepsilon_s}{\mu_s}}\cos\alpha$, $B^p = \frac{\omega}{c}\sqrt{\frac{\mu_s}{\varepsilon_s}}\left(\mu(z)\varepsilon(z) - \mu_s\varepsilon_s\sin^2\alpha\right)$, α is the angle of incidence, $\beta(z)$ is the angle of refraction, which is related to the initial angle of incidence α via the Snell's law $\sqrt{\varepsilon(z)\,\mu(z)}\sin\beta(z) = \sqrt{\varepsilon_s\,\mu_s}\sin\alpha$, and $\varepsilon(z)$ and $\mu(z)$ are the dielectric permittivity and magnetic permeabilities of the inhomogeneous medium, and ε_s and μ_s are the dielectric and magnetic permeabilities of the medium bordering on both sides of the inhomogeneous medium layer, $k_{0z} = \frac{\omega}{c}\sqrt{\varepsilon_s\,\mu_s}\cos\alpha$, $\omega = 2\pi\nu$, ν is the frequency of the incident light, c is the speed of light in vacuum, L is the inhomogeneous layer thickness.

The total field inside 1D inhomogeneous layer $E^{s,p}(z)$ are determined by equation [15]

$$E^{s,p}(z) = \frac{k_0}{k_L}\left[F_{-k}^{s,p}(z) + R^{s,p}(L)\,F_k^{s,p}(z)\right]E_i^{s,p}.$$

Here and above, the indices k and $-k$ denote the functions computed before and after the inversion, $A^{s,p} \to -A^{s,p}$ and $B^{s,p} \to -B^{s,p}$ in system (1).

Finally, coefficients of reflection and transmission and light intensity inside the layer are determined by the expressions: $T^{s,p} = [t^{s,p}]^2$, $R^{s,p} = [r^{s,p}]^2$, $I_{in}^{s,p}(z) = |E^{s,p}(z)|^2$.

As an illustrative example of applying our proposed method, let us consider the problem of scattering of a plane wave incident at an angle α on a photonic crystal (PC) layer with an aperiodic dependence of dielectric permittivity $\varepsilon(z) = \varepsilon_0 + a(z)\cos\left(\frac{2\pi}{\Lambda}z\right)$, namely with layer with a modulation amplitude gradient (apodized grating), here ε_0 is the constant value, and a and Λ are the depth and the period of modulation, correspondingly. It is assumed that the layer is sandwiched between two media with permittivities ε_s. Fig. 1

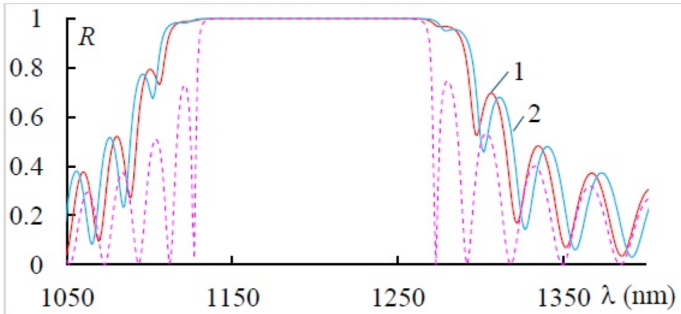

Figure 1: The spectra of the reflection R at different laws of apodization. The parameters are: $\varepsilon_0 = 2.25$, $\Lambda = 400nm$, PC layer thickness, $d = 16\mu m$, $a_{\min} = 0.3$, $a_{\max} = 0.8$, and $a(z) = const = 0.5$ for the case of absence of anodization (dashed line).

shows the spectra of the reflection R at different laws of apodization and also at absence of apodization, that is at $a(z) = const$. We consider two types of apodized PCs, namely, the first type with $a(z) = \frac{a_{\max} - a_{min}}{d} z + a_{\min}$ (curve 1) and the second type $a(z) = \frac{a_{\min} - a_{max}}{d} z + a_{\max}$ (curve 2). In the first case, along the direction of light propagation, the modulation depth increases linearly from the value of $a_{\max}$ at the input surface to the value of $a_{\min}$ at the output surface, and in the second case, it decreases linearly from the value of $a_{\max}$ at the input surface to the value of $a_{\min}$ at the output surface.

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THEORY OF DEFECT MODES MERGING IN ONE-DIMENSIONAL PHOTONIC CRYSTALS WITH TWO DEFECTS

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The majority of researchers investigating one-dimensional photonic crystals (1D PCs) and sensors based on them consider structures with a single defect layer (DL) in the center of them due to the simplicity of their analysis. However, it seems to us that structures with several DLs have a number of potential advantages and their investigation is very promising. The presence of several defect modes (DMs) in the spectrum allows extending the application possibilities of such PCs. One of the obvious applications of such structures is their promising potential as optical filters, due to the possibility of manipulation of two or more DMs independently by varying the number and thickness of DLs in the PC structure [1, 2]. Other researchers have noted the following perspectives of application of 1D PCs with two DLs in the structure. In [3,4], the advantages of 1D PCs with two DLs for fabrication of ultrafast all-optical switching devices and for broadband energy localization under the action of elastic waves were shown. Finally, researches [5, 6] demonstrated the promising potential of one-dimensional dual-defect PCs for the development of temperature sensors and sensors for biomedical applications. These works also highlight the importance of the investigation of structures with two DLs.

One of the interesting features of PCs with two DLs is the merging of two DMs on the reflection spectrum at certain structure parameters. Some cases of DM merging have been considered in certain works [4, 7, 8], but there has been no comprehensive investigation of this topic and only the case of mode merging at a perfectly reflecting central mirror of the PC has been considered. The investigation of the conditions under which these modes merge, as well as the analysis of the various types of merging, is of great significance, because merged modes may demonstrate unusual properties, for example, their sensitivity as optical sensors may be higher than that in similar structures with a single DL. Therefore, for the first time, we have obtained analytical expressions for determination of the wavelength of DMs in 1D PCs with two DLs from the condition of zero reflection. Analytical formulas on the conditions for the merging of two DMs and on the DMs with zero value of the reflection coefficient have been obtained. We also have considered different types of DMs merging (intersection, touching and broadband merging) and analytically derived the conditions for each of them.

We have obtained following analytical expression for the changes of phase of the wave at a single passage through each of the DLs (φ_1 and φ_2) in 1D PCs with two DLs from the condition of zero reflection: $\left(e^{2i\varphi_1} - \frac{\tilde{r}_I^*}{r_{II}}|r_{II}|^2\right)\left(e^{2i\varphi_2} - \frac{\tilde{r}_{II}^*}{r_{III}}\right) + \frac{\tilde{r}_I^*}{r_{II}}\frac{\tilde{r}_{II}^*}{r_{III}}\left(1-|r_{II}|^2\right) = 0$, where r and t are the reflection and transmission coefficients of the each cavities in the structure, $\tilde{r}_I^* = -r_I t_I^*/t_I$, $\tilde{r}_{II}^* = -r_{II} t_{II}^*/t_{II}$. From this equation on defect phases, we have expressed the solution for the sum of phases $\varphi_m = \varphi_1 + \varphi_2$ through the phase difference $\Delta\varphi = \varphi_1 - \varphi_2$:

$$\varphi_{m_{1,2}} = -i \cdot \ln\left[\frac{1}{2}\left(\frac{\tilde{r}_{II}^*}{r_{III}}e^{i\Delta\varphi} + \frac{\tilde{r}_I^*}{r_{II}}e^{-i\Delta\varphi}|r_{II}|^2 \pm \sqrt{\left(\frac{\tilde{r}_{II}^*}{r_{III}}e^{i\Delta\varphi} + \frac{\tilde{r}_I^*}{r_{II}}e^{-i\Delta\varphi}|r_{II}|^2\right)^2 - \frac{4\tilde{r}_I^*\tilde{r}_{II}^*}{r_{II}\,r_{III}}}\right)\right] + 2\pi q_1, \quad (1)$$

where $q_1 \in \mathbb{Z}$.

From this equation we have obtained a condition on the touching of the phase curves at which DMs merge on the reflection spectrum:

$$\Delta\varphi = \frac{1}{2}\left(\tilde{\rho}_{II} - \rho_{II} + \rho_{III} - \tilde{\rho}_I\right) + \frac{1}{2}\,i \cdot \ln\left[\frac{2 - |r_{II}|^2 \pm 2\sqrt{1-|r_{II}|^2}}{|r_{II}|^2 \cdot |r_{III}| \cdot |r_I|}\right] + \pi q_2, \quad (2)$$

where $q_2 \in \mathbb{Z}$, $\rho = \mathrm{Arg}(r)$ is the phase of reflection coefficients: $r = |r|e^{i\rho}$. We can obtain the condition on the wide-band merging of DMs:

$$\begin{cases} \xi > & \dfrac{2-|r_{II}|^2 - 2\sqrt{1-|r_{II}|^2}}{|r_{II}|^2|r_{III}||r_I|} \\ \mathrm{Re}[\Delta\varphi] = & \dfrac{2\pi}{\lambda}(n_{d1}d_{d1} - n_{d2}d_{d2}) \end{cases}. \quad (3)$$

This system of equations must be satisfied over the whole wavelength range in which broadband merging is to be obtained. The analytical equations (2) and (3) obtained in this study permit the calculation of the parameters of the PC at which DMs merge at the given wavelength or within the specified range of wavelengths.

Figure 1 shows examples of different types of DMs merging on the reflection spectrum. The parameters for the structures are found from the solution of Eq. (2) and (3). This figure shows three types of DMs merging: touching (a), intersection (b) and broadband merging (c).

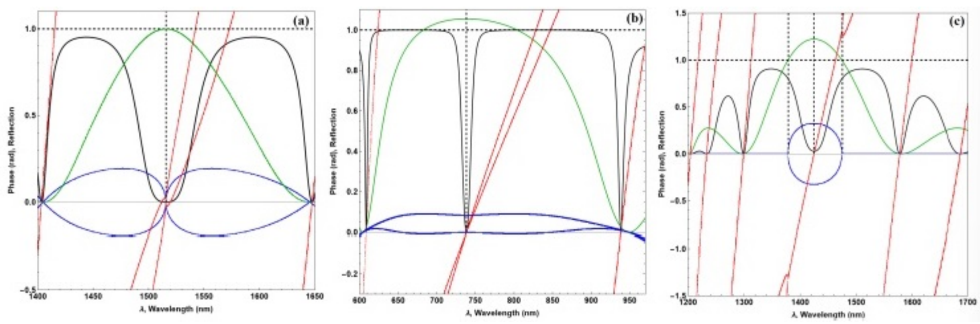

Figure 1: Spectra of Re[φ_m] (red), Im[φ_m] (blue), ξ (green) and reflection spectrum $|R|^2$ (black) for structure ($AB|D1|AB|D2|BA$) (a), ($BA|D1|AB|D2|BA$) (b) and $AB|D1B|AB|D2|BA$ (c). The parameters of the structures are $N_1 = N_3 = 9$, $N_2 = 14$, $n_1 = 1.8$, $n_2 = 2.0$, $d_1 = 210.5nm$, $d_2 = 189.5nm$, $d_{d1} = 248.6nm$, $d_{d2} = 0$, $n_{d1} = n_{d2} = 1.52$ (a); $N_1 = 2$, $N_2 = 6$, $N_3 = 4$, $n_1 = 1.5$, $n_2 = 2.4$, $d_1 = 123nm$, $d_2 = 77nm$, $d_{d1} = 0$, $d_{d2} = 277.6nm$, $n_{d1} = n_{d2} = 1.33$ (b); $N_1 = N_3 = 6$, $N_2 = 10$, $n_1 = 1.34$, $n_2 = 1.52$, $d_1 = 265.5nm$, $d_2 = 234.5nm$, $d_{d1} = 536nm$, $d_{d2} = 536nm$, $n_{d1} = n_{d2} = 1.33$ (c). The normal incidence of light on the PC was considered.

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ON ORTHOGONAL WEB $W(4,3,2)$ ASSOCIATED WITH CURVILINEAR WEB $W(4,3,1)$

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Let a curvilinear 4-web $W(4,3,1)$ be defined on a differentiable manifold M^3. Its structural equations have the form

$$d\omega^i = \omega^i \wedge \omega + 3a^{i[1}\omega^2 \wedge \omega^{3]}, \quad d\omega = 3b^{[1}\omega^2 \wedge \omega^{3]}, \quad \nabla a^{ij} = a^{ij}{}_k \omega^k \qquad (i = 1,2,3),$$

where the fundamental tensors of the web satisfy the relations [1]:

$$a^{ij} = a^{ji}, \quad a^{ij}{}_k = a^{ji}{}_k, \quad b^i = a^{ij}{}_j.$$

The symmetric tensor a^{ij} defines an invariant cone $\psi = a^{ij}\xi_i\xi_j$ of the second order in the tangent space $T_x(M^3)$ of the manifold M^3, where ξ_i are the tangential coordinates of the two-dimensional subspace P_x^2 of the space $T_x(M^3)$, and the equation $\psi = 0$ defines a curve Q of the second order in tangential coordinates ([P_x^2 is the projectivization of the space $T_x(M^3)$).

Let an equiangular metric be defined on the manifold M^3 carrying the web $W(4,3,1)$ [2, pp. 52–54], with metric tensor

$$(g_{ij}) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

A non-degenerate differential quadratic form $\varphi = 3\sum_{i=1}^{3}(\omega^i)^2 - 2\sum_{i<j}\omega^i\omega^j$ defines a conformal structure on the manifold M^3 [3].

On a manifold M^3 carrying the web $W(4,3,1)$, we consider two-dimensional distributions Δ_α^2 ($\alpha = 0,1,2,3$) which are orthogonal to the web lines $\lambda_\alpha(x)$ with respect to the above-mentioned equiangular metric. If the distributions Δ_α^2 are integrable, then they define on M^3 a 4-web $W(4,3,2)$ of codimension 1, formed by four families of two-dimensional surfaces. This web will be called orthogonal to the original web. It will be a nonholonomic associated web [4] of the curvilinear web $W(4,3,1)$. The following statement and theorems are true:

Statement. *Two-dimensional distributions are completely integrable if and only if the following relations are satisfied:* $a^{11} = a^{12} + a^{31}$, $a^{22} = a^{23} + a^{12}$, $a^{33} = a^{31} + a^{23}$, $a^{12} + a^{23} + a^{31} = 0$.

Theorem 1. *The orthogonal web* $W(4,3,2)$ *is octahedral if and only if the associated curved web* $W(4,3,1)$ *is parallelizable.*

Theorem 2. *For the orthogonal web* $W(4,3,2)$ *to be hexagonal, it is necessary and sufficient that the affine connection* γ *induced by the associated curvilinear web* $W(4,3,1)$ *have absolute parallelism.*

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ON THE INTERNAL STABILITY NUMBER OF A GRAPH

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The number of internal stability (independence) of a graph is one of its main characteristics. Therefore, knowledge of its value is of certain interest. In addition, it is used in solving other problems, for example, when considering the problem of correctly coloring the vertices of a graph. Let G be a graph. A set of vertices V in G is called internally stable if no two vertices in V are adjacent. The internal stability number of a graph G is the quantity $\alpha(G) = \max|V|$, where the maximum is taken over all internal stable sets V, and $|V|$ means the cardinality of the set V.

Let $G(n)$ denote the set of all undirected graphs with n vertices numbered $1, 2, \ldots, n$, and let E be an arbitrary property of graphs. Let $G(n,E)$ denote the subset of all graphs from $G(n)$ that have the property E. Almost all graphs are said to have property E if $\lim\limits_{n\to\infty} \frac{|G(n,E)|}{|G(n)|} = 1$. In this note we find the value of the internal stability number for almost all graphs of the set $G(n)$. The following theorem is true:

Theorem. *For almost all graphs in* $G(n)$*, the internal stability number* $\alpha(G)$ *is either* $[2(\log_2 n - \log_2 \log_2 n)]$ *or* $[2(\log_2 n - \log_2 \log_2 n)] + 1$.