VIIth Summer School

Algebra, Topology, and Analysis

July 5 — July 16, 2010
Verkhovyna, Ukraine

VII-та Літня школа

Алгебра, топологія і аналіз

5 липня — 16 липня, 2010
Верховина, Україна

Organizers of the Summer School

Організатори Літньої школи

Vasyl' Stefanyk Precarpathian National University, Ivano-Frankivsk

Прикарпатський національний університет імені Василя Стефаника, Івано-Франківськ

Ivan Franko Lviv National University, Lviv

Львівський національний університет імені Івана Франка, Львів

Abstracts of Lectures and Reports

Тези лекцій і доповідей

ON THE COMPLETENESS FOR THE SYSTEMS OF DIFFERENTIAL EQUATIONS

A.V. Agibalova

Donetsk National University, Donetsk, University Street 24, Ukraine
E-mail address: agannette@rambler.ru

Consider in $L^2([0,1];\mathbb{C}^n) := L^2[0,1] \otimes \mathbb{C}^n$ the first-order systems of ordinary differential equations

$$\frac{1}{i} B \frac{dy}{dx} + Q(x)y = \lambda y, \qquad y = \mathrm{col}(y_1, ..., y_n) \tag{1}$$

with the nondegenerate diagonal $n \times n$ matrix

$$B = \mathrm{diag}(b_1^{-1} I_{n_1}, \ldots, b_r^{-1} I_{n_r}), \quad n = n_1 + \ldots + n_r,$$

where $b_j \neq b_k$ for $j \neq k$, $Q(\cdot)$ the summable potential matrix, i. e. $Q(\cdot) \in L^1([0,1];\mathbb{C}^n)$, $Q = (Q_{jk})_{j,k=1}^r$ is its block-matrix representation with respect to the orthogonal decomposition $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \ldots \oplus \mathbb{C}^{n_r}$.

Systems (1) are of significant interest in some theoretical and practical questions. For example, if $n = 2m$, $r = 2$, $B = \mathrm{diag}(I_m, -I_m)$ and $Q_{11} = Q_{22} = 0$, then the system (1) is equivalent to the Dirac system (see [3]). For $r = n$ and $b_j = e^{2\pi i j/n}$, an nth-order differential equation is reduced to the system (1).

We consider the 2×2 Dirac type system

$$-iBy' + Q(x)y = \lambda y, \qquad y = \mathrm{col}(y_1, y_2), \qquad x \in [0,1], \tag{2}$$

where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \in \mathbb{C}\backslash\mathbb{R}, \quad \text{and} \tag{3}$$

$$Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix}, \quad Q_{12}(x), Q_{21}(x) \in L_1[0,1].$$

To the system (2) we attach boundary conditions of the form

$$\begin{aligned} U_1(y) &:= y_1(0) = 0, \\ U_2(y) &:= a_{22}y_2(0) + a_{23}y_1(1) + a_{24}y_2(1) = 0. \end{aligned} \tag{4}$$

The following theorem complement some results from [2].

. *Let $Q_{21}(\cdot) \in C[0,1]$. If $a_{22}a_{23}a_{24} \neq 0$ and $Q_{21}(1) \neq 0$, then the system of root vectors of the problem* (2)–(4) *is complete in $L_2\left([0,1];\mathbb{C}^2\right)$.*

The talk is based on joint work with M. M. Malamud and L. L. Oridoroga.

[1] M. M. Malamud, *On the completeness of the system of root vectors of Sturm-Liouville operator subject to general boundary conditions*, Func. Analysis and its Appl. **42**(3) (2008), 45–52

[2] M. M. Malamud, L. L. Oridoroga, *Completeness theorems for systems of differential equations*, Func. Analysis and its Appl. **34**(4) (2000), 88–90

[3] V. A. Marchenko, Sturm-Liouville Operators and Their Applications, Kyiv, Naukova Dumka, 1977

AN ELEMENT OF STABLE RANGE 1 AND A RING OF AN ALMOST STABLE RANGE 1

S. I. Bilavska

Department of Algebra and Logic, Ivan Franko National University of Lviv, Universytetska street 1, 79000, Ukraine
E-mail address: `zosia _ meliss@yahoo.co.uk`

Let R is a commutative ring with $1 \neq 0$.

Definition 1. *Note, that a row* $(a_1, a_2, \dots a_n) \in R^n$ *is an unimodular, if* $a_1R + a_2R + \ldots + a_nR = R$*, that is, exist* $u_1, u_2, \dots u_n \in R$ *such that* $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 1$.

Definition 2. *The smallest positive natural* n *called a stable rank* n *of a ring* R *if performed: for any unimodular row* $(a_1, \dots a_n, a_{n+1})$ *length* $n+1$ *exist an elements* $b_1, b_2, \dots b_n \in R$ *such that a row* $(a_1 + a_{n+1}b_1, a_2 + a_{n+1}b_2, \dots a_n + a_{n+1}b_n)$ *is a unimodular. We denote it by* $st.r(R) = n$. $[1-2]$

Let consider it more detail: if $n = 1$, then for a unimodular row (a, b) exists $t \in R$ such that $a + bt$ is an invertible element [3]. If $n = 2$, then for a unimodular row (a, b, c) exist $x, y \in R$ such that $(a + cx, b + cy)$ is unimodular.[2]

Definition 3. *Element* $a \in R$ *called element of a stable range 1, if for any* $b \in R$ *exists* $t \in R$*, such that* $a + bt$ *is an invertible element of a ring* R.

1. *Let* R *is a commutative ring. Then any idempotent* $e \in R$ *is an element of a stable range 1.*

Definition 4. *Commutative ring* R *is Bezout ring if every finitely generated ideal of ring* R *is a principal.*

2. *Let* R *is a commutative Bezout ring. Then a set of element of stable range 1 is a multiplicative closed.*

Definition 5. *Element* a *of a ring* R *called element of almost stable range 1, if* $st.r(R/aR) = 1$.

Definition 6. *Ring* R *is a ring of an almost stable range 1 if for any ideal* I, $I \nsubseteq J(R)$, $st.r(R/I) = 1$*, where* $J(R)$ *is Jacobson radical .*

1. *Let* R *is a ring of almost stable range 1, then any unimodular row over* R *supplemented with invertible matrice.*

3. *Let a is an element of an almost stable range 1 of a commutative ring R. If $aR + bR + cR = R$, then exist element $y \in R$ such that $aR + (b + cy)R = R$.*

4. *Let a is an arbitrary element of a ring R, such that for any $b, c \in R$, $aR + bR + cR = R$ and exists $y \in R$ such that $aR + (b + cy)R = R$. Then a is an element of almost stable range 1.*

2. *Let R is a ring in which every non zero and non invertible element is an element of an almost stable range 1 and if $J(R) \neq 0$, then $st.r(R) = 1$.*

3. *Let R is Bezout ring in which any element is an element of an almost stable range 1. Then for any square matrice A, $detA \neq 0$, size $n \times n$ over R exist matrices $P \in GE_n(R)$ and $Q \in GL_n(R)$ such that*

$$PAQ = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix},$$

where ε_i is elementary divisor of matrice A, $1 \leq i \leq n$. [4]

Note, that $GL_n(R)$ - group of invertible matrice over ring R.
$GE_n(R)$ - subgroup of $GL_n(R)$ generated of elementary matrices.

[1] H. Bass *K-theory and stable algebra*, Inst. Hautes Etudes. Sci.Publ. Math., **22** (1964), 485–544.

[2] L.N. Vaserstein *The stable rank of ring and dimensionality of topological spaces*, Functional Anal. Appl., **5** (1971), 102–110.

[3] L.N. Vaserstein *Bass's first stable range condition*, J. Pure and Appl. Alg., **34** (1984), 319–330.

[4] B.V. Zabavsky *Diagonalizability theorem for matrices over rings with finite stable range*, Alg.Discr.Math. - 2005. - N1 - 134–148.

EXTENT, NORMALITY AND OTHER PROPERTIES OF SPACES OF SCATTEREDLY CONTINUOUS MAPS

B.M. Bokalo and N.M. Kolos

Department of Mechanics and Mathematics, Ivan Franko National University, Lviv, Universytetska street 1, Ukraine
E-mail address: Bogdanbokalo@mail.ru
E-mail address: Nadiya_Kolos@ukr.net

A map $f : X \to Y$ between topological spaces is called scatteredly continuous if for each non-empty subspace $A \subset X$ the restriction $f|_A$ has a point of continuity.

We study properties of scatteredly continuous maps between topological spaces and properties of topological spaces of scatteredly continuous maps. In particular, we will talk about normality and extent of spaces of scatteredly continuous maps.

[1] R. Engelking, General Topology, PWN, Warzawa, 1977.
[2] B. Bokalo, N. Kolos, *When does $SC_p(X) = \mathbb{R}^X$ hold?*, Topology, Vol.**48**(2009), 178-181.
[3] Arkhangel'skii A.V., Topological spaces of functions, M.: MGU, 1989 (in Russian).
[4] Arkhangel'skii A.V., Bokalo B.M., *The tangency of topologies and tangential properties of topological spaces*, Trudy Moskov. Mat. Obshch. **54** (1992), 160-185, 278-279 (in Russian).
[5] T. Banakh, B. Bokalo, *On scatteredly continuous maps between topological spaces*, Topology and Appl., Vol. **157** (2010), 108-122.

ALGEBRAS OF ENTIRE ANALYTIC FUNCTIONS ON ℓ_p

I.V. Chernega

Institute for Applied Problems of Mechanics and Mathematics, Lviv, Naukova Str. 3 b, Ukraine
E-mail address: icherneha@ukr.net

We shall denote by $\mathcal{H}_b(\ell_p)$ the algebra of entire analytic functions of bounded type on ℓ_p and by $\mathcal{H}_{bs}(\ell_p)$ its subalgebra of all symmetric functions. Also we use the notations $M_b(\ell_p)$ and $M_{bs}(\ell_p)$ for spectra of the algebras $\mathcal{H}_b(\ell_p)$ and $\mathcal{H}_{bs}(\ell_p)$ respectively, that is, the set of all non-null continuous complex homomorphisms. In [1] the spectra of algebras of symmetric holomorphic functions on ℓ_p are investigated. Maximal ideals of algebras of analytic functions were studied in [2], [3].

We study the relationship between the spectra of $\mathcal{H}_{bs}(\ell_p)$ and $\mathcal{H}_b(\ell_p)$. If $\varphi \in M_b(\ell_p)$ then the restriction φ^s of φ to $\mathcal{H}_{bs}(\ell_p)$ is a complex homomorphism of $\mathcal{H}_{bs}(\ell_p)$. According to [3] there exists a sequence of Banach spaces $(E_n)_{n=1}^{\infty}$ and a sequence of maps $\delta^{(n)} : E_n \to M_b(\ell_p)$, where $E_1 = \ell_p$, E_n coincides with the subspace of all functionals on $\mathcal{P}({}^n\ell_p)$ which vanish on finite sums of products of polynomials of degree less than n and $\delta^{(1)}(z)(f) = f(z)$, such that for every $\varphi \in M_b(\ell_p)$

$$\varphi(f) = *_{n=1}^{\infty}\delta^{(n)}(u_n)(f) \tag{1}$$

for some $u_n \in E_n$, $n = 1, 2, \ldots$ and the convolution operation " $*$ " for elements $\varphi, \theta \in M_b(\ell_p)$ is defined by

$$(\varphi * \theta)(f) = \varphi(\theta(f(\cdot + x))), \text{ where } f \in \mathcal{H}_b(X). \tag{2}$$

Hence for every $\varphi \in M_b(\ell_p)$, φ^s has the representation

$$\varphi^s = \Big(*_{n=1}^{\infty}\, \delta^{(n)}(u_n)\Big)^s.$$

Can we extend this formula for an arbitrary complex homomorphism of $\mathcal{H}_{bs}(\ell_p)$? Clearly, it is so if we can extend each character in $M_{bs}(\ell_p)$ to a character in $M_b(\ell_p)$.

. *If there exists a continuous homomorphism $\Phi : \mathcal{H}_b(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$, then every character $\theta \in M_{bs}(\ell_p)$ can be extended to a character $\varphi \in M_b(\ell_p)$ by the formula $\varphi(f) = \theta(\Phi(f))$. Moreover, if Φ is a projection then $\varphi^s = \theta$.*

We study the existence of a homomorphism from $\mathcal{H}_b(\ell_p)$ onto $\mathcal{H}_{bs}(\ell_p)$ and conditions of its continuity.

[1] R. Alencar, R. Aron, P. Galindo, and A. Zagorodnyuk, *Algebras of symmetric holomorphic functions on ℓ_p*, Bull. Lond. Math. Soc. **35** (2003), 55–64

[2] R.M. Aron, B.J. Cole, and T.W. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. Reine Angew. Math. **415** (1991), 51–93

[3] A. Zagorodnyuk, *Spectra of algebras of entire functions on Banach spaces*, Proc. Amer. Math. Soc. **134** (2006), 2559–2569

TOPOLOGICAL INVERSE MONOIDS OF ALMOST MONOTONE INJECTIVE CO-FINITE PARTIAL SELFMAPS OF POSITIVE INTEGERS

Ivan Chuchman and Oleg Gutik

Department of Mechanics and Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine

E-mail address: chuchman_i@mail.ru

E-mail address: o_gutik@franko.lviv.ua, ovgutik@yahoo.com

In this paper all spaces are assumed to be Hausdorff. Furthermore we shall follow the terminology of [1, 4, 5].

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists the unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion.*

A *semitopological* (resp. *topological*) *semigroup* is a topological space together with a separately (resp. jointly) continuous semigroup operation. A *topological inverse semigroup* is an inverse topological semigroup with the continuous inversion.

Let $\mathbb{N}$ be the set of all positive integers. A partial map $\alpha\colon \mathbb{N} \rightharpoonup \mathbb{N}$ is called *almost monotone* if there exists a finite subset A of $\mathbb{N}$ such that the restriction $\alpha|_{\mathbb{N}\setminus A}\colon \mathbb{N}\setminus A \rightharpoonup \mathbb{N}$ is a monotone partial map. By $\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N})$ we shall denote the semigroup of monotone, almost non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N}\setminus\operatorname{dom}\varphi$ and $\mathbb{N}\setminus\operatorname{rank}\varphi$ are finite for all $\varphi \in \mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N})$.

Chuchman and Gutik showed that every Hausdorff Baire topology τ on $\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N})$ such that $\left(\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N}),\tau\right)$ is a semitopological semigroup is discrete [2, **?**].

We construct two non-discrete (and hence non-Baire) topologies τ_1 and τ_2 on the semigroup $\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N})$ such that the following assertions hold:

(*i*) $\left(\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N}),\tau_1\right)$ is a topological inverse semigroup and every $\mathscr{H}$-class in $\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N})$ is an open-and-closed subset of $\left(\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N}),\tau_1\right)$;

(*ii*) $\left(\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N}),\tau_2\right)$ is a topological inverse semigroup and every $\mathscr{H}$-class in $\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N})$ is a closed non-open subset of $\left(\mathscr{I}_{\infty}^{\,\vec{\nearrow}}(\mathbb{N}),\tau_1\right)$.

[1] J. H. Carruth, J. A. Hildebrant and R. J. Koch, *The Theory of Topological Semigroups*, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.
[2] I. Chuchman and O. Gutik, *Topological monoids of almost monotone injective cofinite partial selfmaps of positive integers*, Conference on complex analysis dedicated to the memory of A. A. Goldberg (1930–2008). Lviv, Ukraine, May 31-June 5, 2010. Abstracts. Lviv, 2010, P. 8–9.
[3] I. Chuchman and O. Gutik, *Topological monoids of almost monotone, injective cofinite partial selfmaps of positive integers*, Preprint.
[4] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.
[5] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.

SUPEREXTENSIONS OF SEMILATTICES

Volodymyr Gavrylkiv

Department of Mathematics and Computer Sciences, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Shevchenko Street 57, Ukraine
E-mail address: vgavrylkiv@yahoo.com

In the talk we describe the algebraic structure of the semigroups $G(X)$, $\lambda(X)$, $N_k(X)$, $Fil(X)$ and $\beta(X)$ over semilattice X (see [5], [6]). The semigroup $G(X)$ ($\lambda(X)$) over group X rarely is commutative: this holds if and only if the group X has finite order $|X| = 1$ ($|X| \leq 4$, see [1]). This leads to the following natural question: are semigroups $G(X)$ or $\lambda(X)$ commutative for some semigroup X of big cardinality $|X|$? We prove that for any finite linear ordered semilattice X the semigroups $G(X)$, $\lambda(X)$, $N_k(X)$, $Fil(X)$ and $\beta(X)$ are commutative semigroups.

[1] T. Banakh, V. Gavrylkiv, O. Nykyforchyn, *Algebra in superextensions of groups, I: zeros and commutativity*, Algebra Discrete Math. **3** (2008), 1-29

[2] T. Banakh, V. Gavrylkiv, *Algebra in superextension of groups, II: cancelativity and centers*, Algebra Discrete Math. **4** (2008), 1-14

[3] T. Banakh, V. Gavrylkiv, *Algebra in the superextensions of groups, III: minimal left ideals*, Mat. Stud. **31**(2) (2009), 142-148

[4] T. Banakh, V. Gavrylkiv, *Extending binary operations to functor-spaces*, Carpathian Mathematical Publication. **1**(2) (2009), 113-126

[5] V. Gavrylkiv, *The spaces of inclusion hyperspaces over noncompact spaces*, Mat. Stud. **28**(1) (2007), 92-110

[6] V. Gavrylkiv, *Right-topological semigroup operations on inclusion hyperspaces*, Mat. Stud. **29**(1) (2008), 18-34

ON SEMITOPOLOGICAL SYMMETRIC INVERSE SEMIGROUPS OF A BOUNDED FINITE RANK

Oleg Gutik and Andriy Reiter

Department of Mechanics and Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine

E-mail address: `o_gutik@franko.lviv.ua, ovgutik@yahoo.com`

E-mail address: `reiter_andriy@yahoo.com, reiter@i.ua`

In this paper all spaces are assumed to be Hausdorff. Furthermore we shall follow the terminology of [1, **?**, **?**, **?**]. By ω we denote the first infinite cardinal.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists the unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

A *semitopological* (resp. *topological*) *semigroup* is a topological space together with a separately (resp. jointly) continuous semigroup operation. A *topological inverse semigroup* is an inverse topological semigroup with the continuous inversion.

Let $\mathscr{I}(X)$ denote the set of all partial one-to-one transformations of X together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \,\text{if}, x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\},$$

for $\alpha, \beta \in \mathscr{I}(X)$.

The semigroup $\mathscr{I}(X)$ is called the *symmetric inverse semigroup* over the set X (see [2]). The symmetric inverse semigroup was introduced by Wagner [13].

We denote $\mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}(X) \mid \operatorname{rank}\alpha \leqslant n\}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathscr{I}_\lambda^n$ $(n = 1, 2, 3, \ldots)$ is an inverse semigroup, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}(X)$, for each $n = 1, 2, 3, \ldots$. We observe that the the symmetric inverse semigroup $\mathscr{I}_\lambda^1$ of finite transformations of the rank 1 is isomorphic to the semigroup of matrix units B_λ.

Let $\mathscr{S}$ be a class of (semi)topological semigroups. A semigroup $S \in \mathscr{S}$ is called *H-closed in $\mathscr{S}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathscr{S}$ which contains S as a subsemigroup [5, **?**]. A (semi)topological semigroup $S \in \mathscr{S}$ is called *absolutely H-closed in the class $\mathscr{S}$* if any continuous homomorphic image of S into $T \in \mathscr{S}$ is H-closed in $\mathscr{S}$ [6, **?**]. A semigroup S is called *algebraically h-closed in $\mathscr{S}$* if S with discrete topology $\mathfrak{d}$ is absolutely H-closed in $\mathscr{S}$ and $(S, \mathfrak{d}) \in \mathscr{S}$ [5].

Gutik and Pavlyk in [7] consider the partial case of the semigroup $\mathscr{I}_\lambda^n$: an infinite topological semigroup of $\lambda \times \lambda$-matrix units B_λ. There they show that an infinite topological semigroup of $\lambda \times \lambda$-matrix units B_λ does not embed into a compact topological semigroup and B_λ is algebraically h-closed in the class of topological inverse semigroups.

Gutik, Lawson and Repovš in [4] introduce the notion of semigroup with a tight ideal series and investigate their closures in semitopological semigroups, particularly inverse semigroups with continuous inversion. As a corollary they show that the symmetric inverse semigroup of finite transformations $\mathscr{I}_\lambda^n$ of infinite cardinal λ is algebraically closed in the class of (semi)topological inverse semigroups with continuous inversion.

In [9] Gutik and Reiter show that the topological inverse semigroup $\mathscr{I}_\lambda^n$ is algebraically h-closed in the class of topological inverse semigroups. Also they prove that a topological semigroup S with countably compact square $S \times S$ does not contain the semigroup $\mathscr{I}_\lambda^n$ for infinite cardinal λ and show that the Bohr compactification of an infinite topological semigroup $\mathscr{I}_\lambda^n$ is the trivial semigroup.

In [8] Gutik, Pavlyk and Reiter show that a topological semigroup of finite partial bijections $\mathscr{I}_\lambda^n$ of infinite set with a compact subsemigroup of idempotents is absolutely H-closed and any countably compact topological semigroup does not contain $\mathscr{I}_\lambda^n$ as a subsemigroup. Also they give sufficient conditions onto a topological semigroup $\mathscr{I}_\lambda^1$ to be non-H-closed.

1. *The semigroup $\mathscr{I}_\lambda^n$ is algebraically h-closed in the class of semitopological inverse semigroups with continuous inversion.*

We describe all congruences on the semigroup $\mathscr{I}_\lambda^n$ and construct a Hausdorff compact topology τ_c on $\mathscr{I}_\lambda^n$ such that $(\mathscr{I}_\lambda^n, \tau_c)$ is a semitopological inverse semigroup with continuous inversion.

2. *Let $\lambda \geqslant \omega$, $n = 1, 2, 3, \ldots$, and τ be a Hausdorff topology on the semigroup $\mathscr{I}_\lambda^n$. Then the following conditions are equivalent:*

(i) $(\mathscr{I}_\lambda^n, \tau)$ is a compact semitopological semigroup;
(ii) $(\mathscr{I}_\lambda^n, \tau)$ is topologically isomorphic to $(\mathscr{I}_\lambda^n, \tau_c)$;
(iii) $(\mathscr{I}_\lambda^n, \tau)$ is a countably compact semitopological semigroup;
(iv) $(\mathscr{I}_\lambda^n, \tau)$ is a countably compact semitopological semigroup with continuous inversion.

[1] J. H. Carruth, J. A. Hildebrant and R. J. Koch, *The Theory of Topological Semigroups*, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.

[3] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.

[4] O. Gutik, J. Lawson, and D. Repovš, *Semigroup closures of finite rank symmetric inverse semigroups*, Semigroup Forum **78**:2 (2009), 326–336.

[5] O. V. Gutik and K. P. Pavlyk, *H-closed topological semigroups and Brandt λ-extensions*, Mat. Metody Phis.-Mech. Polya. **44**:3 (2001), 20–28 (in Ukrainian).

[6] O. V. Gutik and K. P. Pavlyk, *Topological Brandt λ-extensions of absolutely H-closed topological inverse semigroups*, Visnyk Lviv Univ. Ser. Mech.-Math. **61** (2003), 98–105.

[7] O. V. Gutik and K. P. Pavlyk, *On topological semigroups of matrix units*, Semigroup Forum **71**:3 (2005), 389–400.

[8] O. Gutik, K. Pavlyk and A. Reiter, *Topological semigroups of matrix units and countably compact Brandt λ^0-extensions*, Mat. Stud. **32**:2 (2009), 115–131.

[9] O. V. Gutik and A. R. Reiter, *Symmetric inverse topological semigroups of finite rank $\leqslant n$*, Mat. Metody Phis.-Mech. Polya. **53**:3 (2009) 7–14.

[10] W. Ruppert, *Compact Semitopological Semigroups: An Intrinsic Theory*, Lecture Notes in Mathematics, Vol. 1079, Springer, Berlin, 1984.

[11] J. W. Stepp, *A note on maximal locally compact semigroups,* Proc. Amer. Math. Soc. **20**:1 (1969), 251–253.
[12] J. W. Stepp, *Algebraic maximal semilattices,* Pacific J. Math. **58**:1 (1975), 243–248.
[13] V. V. Wagner, *Generalized groups*, Dokl. Akad. Nauk SSSR **84** (1952), 1119—1122 (in Russian).

ON NON-NEGATIVE INTEGER QUADRATIC FORMS

G.V. Kriukova

Department of Algebra and Mathematical logic, The Faculty of Mechanics and Mathematics, National Taras Shevchenko University of Kyiv, Kyiv, Volodymyrska 64, Ukraine
E-mail address: galyna.kriukova@gmail.com

The use of quadratic forms as a tool for characterizing classes of finite dimensional algebras and Lie algebras is well known and widely accepted. We study properties of non-negative integer quadratic forms.

According to Roiter an integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$

$$q(x) = \sum_{i\in\{1,\dots,n\}} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j, \quad (q_i, q_{ij} \in \mathbb{Z})$$

is called *semi integer* if $q_{ij} \in q_i\mathbb{Z}$ for all $i, j \in \{1, \dots, n\}$, and it is called *integer* if in addition $q_i \neq 0$ for all $i \in \{1, \dots, n\}$. The integer form q is called *unit* if $q_i = 1$ for all $i \in \{1, \dots, n\}$. Two forms q and q' and corresponding bigraphs B and B' are *equivalent* if one comes from another due to sequence of sing-invertions. Form is *balanced* if $\forall v \in \mathbb{Z}^n$ such that $q(v) = 0$ holds:

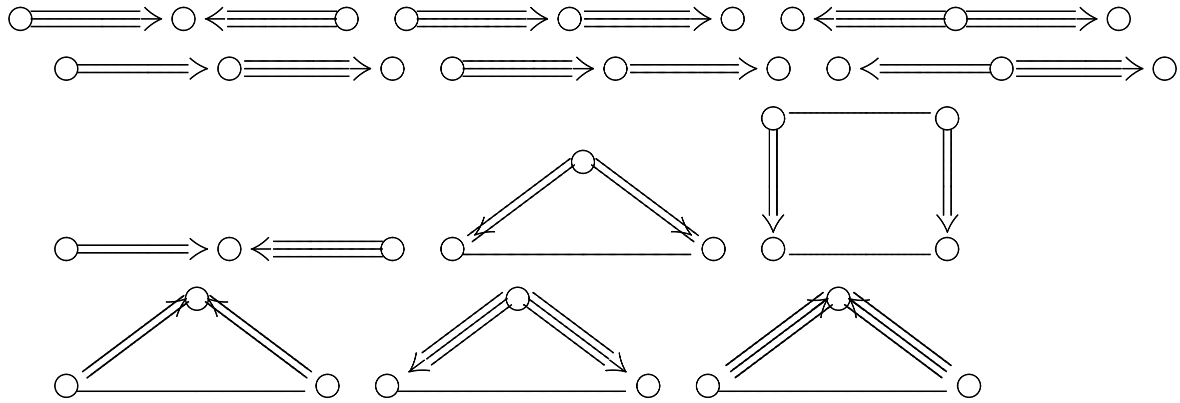
$$(v, y)_q = q(v + y) - q(v) - q(y) = 0, \quad \forall y \in \mathbb{Z}^n.$$

With any such form in n variables one associates its *Coxeter graph* or bigraph B_q, which is labeled and partially directed.

1. *A semi-integer quadratic form q is non-negative iff conditions hold:*

(1) *form q is balanced;*

(2) $q_i \geq 0$, $i \in \{1, \ldots, n\}$;
(3) $q_{ij}^2 \leq 4q_i q_j$, $i, j \in \{1, \ldots, n\}$, $i < j$;
(4) *q does not contain as subform any of form equivalent to following bigraphs:*



This criterion generalizes result of [1] for unit forms. We compare non-negativity criterions for integer quadratic forms, integer unit forms, real quadratic forms ([2]).

[1] M. Barot, J. A. de la Pẽna. *The Dynkin type of a non-negative unit form*, Expositiones Mathematicae. 17 (1999), 339–348.
[2] N.S. Golovaschuk, G.V. Kriukova. *Non-negativity criterion for integer quadratic forms*, Bulletin of University of Kyiv. Series: Physics & Mathematics. 4 (2009)

ON WEAK FILTER CONVERGENCE OF UNBOUNDED SEQUENCES

Alexander Leonov

Department of Mechanics and Mathematics, Kharkov National, University, pl. Svobody 4, 61077 Kharkov, Ukraine
E-mail address: `aleon7@i.ua`

It is known that the properties of sequences that are filter convergent in the weak topology differ significantly from the properties of the ordinary weakly convergent sequences. In particular a weakly convergent sequence must be bounded but, say, a weakly statistically convergent sequence can tend to infinity in norm [1]. This effect induces the following natural question:

. *If a sequence has a weak limit with respect to a given filter $\mathcal{F}$, how quick can the norms of the elements in the sequence tend to infinity?*

Of course the answer depends on the filter. In [3] we prove that For every weakly statistically convergent sequence x_n with increasing norms in a Hilbert space we prove that $\sup_n \|x_n\|/\sqrt{n} < \infty$. This estimate is sharp. We study analogous problem for some other types of weak filter convergence.

[1] J.Connor, M.Ganichev and V.Kadets. *A characterization of Banach spaces with separable duals via weak statistical convergence.* J. Math. Anal. Appl. **244** (2000), no 1, 251 - 261.

[2] V. Kadets. *Weak cluster points of a sequence and coverings by cylinders* / Mat. Fiz. Anal. Geom., **11** (2004), No 2, 161 - 168

[3] V.Kadets, A.Leonov, C.Orhan. *Weak statistical convergence and weak filter convergence for unbounded sequences* J. Math. Anal. Appl. to appear doi:10.1016/j.jmaa.2010.05.031

ON ALGEBRAS OF ULTRADISTRIBUTIONS

V.Ya. Lozynska

Department of Functional Analysis, Pidstryhach Institute of Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, Lviv, 3-b Naukova Str., Ukraine

E-mail address: vlozynska@yahoo.com

The convolution algebras of ultradistributions of Beurling and of Roumieu type are introduced and investigated.

For a weight function ω (see[1]) and an open set $\Omega \in \mathbb{R}^n$ we define

$$\mathcal{E}_{\{\omega\}}(\Omega) = \Big\{ f \in C^\infty(\Omega) | \,\text{for all compact}\, K \in \Omega \text{ there is } m \in \mathbb{N}$$

$$\sup_{\alpha \in \mathbf{N}_0^{\mathbf{N}}} \sup_{x \in K} |f^{(\alpha)}(x)| exp\Big(-\frac{1}{m}\,\varphi^*(m|\alpha|)\Big) < \infty \Big\}$$

and

$$\mathcal{E}_{(\omega)}(\Omega) = \Big\{ f \in C^\infty(\Omega) | \,\text{for all compact}\, K \in \Omega \text{ and all } m \in \mathbb{N}$$

$$p_{K,m}(f) := \sup_{\alpha \in \mathbf{N}_0^{\mathbf{N}}} \sup_{x \in K} |f^{(\alpha)}(x)| exp\Big(-m\varphi^*\Big(\frac{|\alpha|}{m}\Big)\Big) < \infty \Big\},$$

where φ^* denotes the Young conjugate of the convex function φ. We will write $\mathcal{E}_*$ if statement holds for both $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$.

The elements of $\mathcal{E}_{\{\omega\}}(\Omega)'$ (resp. $\mathcal{E}_{(\omega)}(\Omega)'$) are called ultradistributions of Roumieu type (resp. of Beurling type).

For a weight function ω, an ultradistribution $\mu \in \mathcal{E}_*(\mathbb{R}^n)'$, and $f \in \mathcal{E}_*(\mathbb{R}^n)$ we define the convolution by

$$\mu \star f : \mathbb{R}^n \to \mathbb{C}, \quad \mu \star f(t) := \langle \mu_s, f(t+s)\rangle = \langle \mu_s, T_{-s} f(t)\rangle.$$

1. *The space $\mathcal{E}_*(\mathbb{R}^n)'$ is an algebra with respect to the convolution, that is defined by the relation*

$$\mu * \nu : \mathcal{E}_*(\mathbb{R}^n) \to \mathbb{C}, \quad \langle \mu * \nu, f\rangle := \langle \nu, \mu \star f\rangle,$$

$\mu, \nu \in \mathcal{E}_*(\mathbb{R}^n)', f \in \mathcal{E}_*(\mathbb{R}^n)$. *The convolution has the following properties*

$$D^k(\mu \star f) = \mu \star (D^k f) = (-1)^k (D^k \mu) \star f,$$

$$D^k(\mu * \nu) = (D^k \mu) * \nu = \mu * (D^k \nu)$$

for all $k \in \mathbb{Z}_+$.

[1] Braun R.W., Meise R., Taylor B.A., *Ultradifferentiable functions and Fourier analysis* // Results in Mathematics – Vol. 17, (1990) – P. 206–237.

RELATIVELY THIN SUBSETS OF GROUPS

Ie. Lutsenko

Department of Cybernetics, Taras Shevchenko National Kyiv University, Kyiv, Volodimirska 64, Ukraine

E-mail address: ie.lutsenko@gmail.com

Let G be a group with the identity e, $\mathcal{I}$ be a left translation-invariant ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G. A subset $A \subseteq G$ is said to be

- $\mathcal{I}$-*large* if there exist $F \in \mathcal{F}_G$ and $I \in \mathcal{I}$ such that $G = FA \cup I$;
- $\mathcal{I}$-*small* if $L \setminus A$ is $\mathcal{I}$-large for every $\mathcal{I}$-large subset L;
- $\mathcal{I}$-*thin* if $A \cap gA \in \mathcal{I}$ for every $g \in G$, $g \neq e$.

An ideal $\mathcal{I}$ is said to be τ-complete if every $\mathcal{I}$-thin subset of G belong to $\mathcal{I}$.

1. *Let G be an infinite group, $\mathcal{I}$ be a translation-invariant ideal in $\mathcal{P}_G$. Then $\tau(\mathcal{I}) \subseteq \mathcal{S}_{\mathcal{I}}$, where $\mathcal{S}_{\mathcal{I}}$ is the ideal of all $\mathcal{I}$-small subsets of G.*

1. *Let G be an infinite group, $\mathcal{I}$ be a translation-invariant ideal in $\mathcal{P}_G$. Then the ideal $\mathcal{S}_{\mathcal{I}}$ is τ-complete.*

2. *Let $\mathcal{F}$ be a family of subsets of a group G, $A \subseteq G$, $n \in \omega$. Then*

$$A \in \tau^{n+1}(\mathcal{F}) \;\Leftrightarrow \bigcap_{i_0,\dots,i_n \in \{0,1\}} g_0^{i_0} \dots g_n^{i_n} A \in \mathcal{F}.$$

3. *For a group G, the following statements hold*

(1) *G is a Boolean group if and only if $\tau^*(\mathcal{I}_\varnothing) = \tau(\mathcal{I}_\varnothing) = [G]_1$;*
(2) *if G is Boolean then $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$;*
(3) *if G is infinite and $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$ then G is Boolean.*

4. *Let G be an infinite Abelian group with finite subset $\{g \in G : g^2 = e\}$, $\mathcal{T}_G$ be the family of all thin subsets of G, $\mathcal{J}_G$ be the ideal of all sparse subsets of G. Then $\tau(\mathcal{T}_G) \setminus \mathcal{J}_G \neq \varnothing$.*

2. *Let G be an infinite Abelian group with finite number of elements of order 2. Then the ideal $\mathcal{J}_G$ of sparse subsets of G is not τ-complete.*

5. *Let G be a group with no elements of order 2. If $T_1, T_2 \in \mathcal{T}_G$ then $T_1 \cup T_2 \in \tau(\mathcal{T}_G)$.*

6. *Let G be an infinite group of cardinality α, $\mathcal{F}$ be a family of subsets of G closed under taking subsets. Then*

$$\tau*(\mathcal{F}) = \bigcup_{\beta<\alpha^+} \tau^\beta(\mathcal{F}),$$

where $\tau^{\beta+1}(\mathcal{F}) = \tau(\tau^\beta(\mathcal{F}))$ and $\tau^\beta(\mathcal{F}) = \bigcup_{\gamma<\beta} \tau^\gamma(\mathcal{F})$ for a limit ordinal $\beta < \alpha^+$.

ASYMPTOTIC DIMENSION OF SMALL SUBSETS IN COARSE GROUPS

N. Lyaskovska

Department of Geometry and Topology at Ivan Franko National University of Lviv, Ukraine

E-mail address: lyaskovska@yahoo.com

Recall that a subset A of a locally compact group G is

- *large* if there is compact subsets K with $AK = G$.
- *small* if for any large subset L of G the complement $L \setminus A$ is large.

By Th.1.8.11 [1], in the topological space $\mathbb{R}^n$ the ideal of nowhere dense subsets coincides with the ideal of subsets A whose closure has the topological dimension $dim(A) < n$. The following Theorem is an analogue of this fact.

1. *For any discrete finitely generated Abelian group G the subset A is small iff $asdim(A) < asdim(G)$.*

For a subset A of a locally compact group G we write $\operatorname{asdim}(A) \le n$ for an integer number $n \ge 0$ if for every compact subset $K \subset G$ there is compact subset $L \subset G$ and a cover $\mathcal{U}$ of A such that $mesh(\mathcal{U}) \le L$ and $|\{U \in \mathcal{U} : U \cap gK \ne \emptyset\}| \le n+1$ for every $g \in G$. We write $mesh(\mathcal{U}) \le L$ if for any $U \in \mathcal{U}$ there is $g \in G$ with $U \subset gL$. We say that $\operatorname{asdim}(A) = n$ if $\operatorname{asdim}(A) \le n$ and $\operatorname{asdim}(A) \not\le n-1$. If no integer n with $\operatorname{asdim}(A) \le n$ exists, then we put $\operatorname{asdim}(A) = \infty$.

The following examples shows that the Abelian requirement in the previous theorem is essential.

1. *Let F_2 be the free group with two generators a, b. Note that subgroup $A = \{a^n : n \in \mathbb{Z}\}$ is small but has $asdim(A) = asdim(F_2) = 1$.*

2. *For any subset A of a locally compact Abelian group G holds if* $\mathrm{asdim}(A) < asdim(G)\}$ *then A is small.*

[1] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.

VECTOR BUNDLES AND COBORDISMS

Sergiy Maksymenko

Institute of Mathematics of NAS of Ukraine, Kyiv, Tereshchenkivs'ka str., 3, 01601, Ukraine

E-mail address: maks@imath.kiev.ua

Lectures 1-2. Classification of vector bundles. Examples of vector bundles. Regular neighbourhoods of submanifolds. Main constructions over vector bundles: subbundle, factor-bundle, induced bundle, Whitney sum. Embeddings of vector bundles into trivial ones. Vector bundles over $[0, 1]$. Invariance of induced bundles under homotopies. Grassman manifold and the tautological vector bundle. Homotopy classification of vector bundles.

Lectures 3-4. Cobordism theory. The notion of cobordism. Groups of orientable and non-orientable cobordisms. Surgery. Transversality. Thom's construction. The main theorem of cobordism theory (by R. Thom).

ON LAWSON IDEMPOTENT SEMIMODULES

O. Mykytsey

Department of Mathematics and Computer Science, Vasyl' Stefanyk Precarpathian National University, Ivano-Frankivsk, Shevchenka 57, Ukraine

E-mail address: oksana39@i.ua

Let L be a compact Hausdorff Lawson lattice, with $\vee$ and $\wedge$ being resp. join and meet, and let $* : L \times L \to L$ be an upper semicontinuous operation, called multiplication, which is associative, distributive w.r.t. $\vee$ in the both variables, and the top element $1 \in L$ is a two-side unit for $*$. It implies that $*$ is isotone in the both variables, hence $\alpha * \beta \leq \alpha \wedge \beta$ for all $\alpha, \beta \in L$. Then $(L, \vee, *)$ is an idempotent semiring [1].

For an idempotent semiring $S = (S, \vee, *, 0, 1)$ a right $S-$semimodule is a set X with operations $\vee : X \times X \to X$ and $* : X \times S \to X$ such that for all $x, y, z \in X$, $\alpha, \beta \in S$:

1) $x \vee y = y \vee x$;

2) $(x \vee y) \vee z = x \vee (y \vee z)$;

3) there is an (obviously unique) element $\bar{0} \in X$ such that $x \vee \bar{0} = x$ for all x;

4) $(x \vee y) * \alpha = (x * \alpha) \vee (y * \alpha)$, $x * (\alpha \vee \beta) = (x * \alpha) \vee (x * \beta)$;

5) $x * (\alpha * \beta) = (x * \alpha) * \beta$;

6) $x * 1 = x$;

7) $x * 0 = \bar{0}$.

We call X a compact Hausdorff Lawson right $(L, \vee, *)$- semimodule [2] if X is an $(L, \vee, *)$-semimodule and carries a compact Hausdorff topology such that the upper semilattice $(X, \vee)$ is a Lawson lattice [3] and $*$ is lower semicontinuous.

We denote by $(L, \vee, *) - \mathcal{L}w\mathcal{SM}od$ the category that consist of all compact Hausdorff Lawson $(L, \vee, *)$-semimodules and all their continuous maps that preserve all suprema and infima and are $*$-uniform. We also denote by $(L, \vee, *) - \mathcal{L}w\mathcal{SM}od_{\uparrow}$ and $(L, \vee, *) - \mathcal{L}w\mathcal{SM}od_{\downarrow}$ the categories with the same objects, but with the classes of morfisms that consist of all join-preserving (hence isotone) $*$-uniform maps such that the preimages of all closed upper (resp. lower) sets are closed.

For a compact Hausdorff Lawson lower semilattice X, the product $X \times L$ is a compact Hausdorff Lawson lower semilattice as well. Let $\exp^L_\Delta X$ be the ordered by inclusion space of all closed subsets $C \subset X \times \tilde{L}$ such that, for all $\alpha, \beta \in L,\ x, y \in X$:

(1) $\alpha \le \beta, x \le y, (y, \beta) \in C$ implies $(x, \alpha) \in C$;

(2) $(x, \alpha), (x, \beta) \in C$ implies $(x, \alpha \vee \beta) \in C$;

(3) $C \supset (X \times \{0\}) \cup (\{\min X\} \times L)$.

It is proved that $\exp^L_\Delta X$ is a compact Hausdorff Lawson $(L, \vee, *)$-semimodule.

Let $\tilde{L}$ be our compact Hausdorff Lawson lattice L but with reverse order.

. *Is* $\exp^{\tilde{L}}_\Delta X$ *a compact Hausdorff Lawson* $(L, \vee, *)$*-semimodule?*

For $\exp^{\tilde{L}}_\Delta X$ the following conventions are valid:

(1') $\alpha \ge \beta, x \le y, (y, \beta) \in C$ implies $(x, \alpha) \in C$;

(2') $(x, \alpha), (x, \beta) \in C$ implies $(x, \alpha \wedge \beta) \in C$;

(3') $C \supset (X \times \{1\}) \cup (\{\min X\} \times L)$.

For each closed $F \subset X \times \tilde{L}$, the set

$$\theta X(F) = \{\ (x, \alpha) \in X \times \tilde{L} \mid x \le \inf(pr_1(A)), \alpha \ge \inf(pr_2(A))$$
$$\text{for some } A \underset{\mathrm{CL}}{\subset} (F \cup (X \times \{1\}) \cup (\{\min X\} \times \tilde{L})), A \ne \emptyset\}$$

is the least element of $\exp^{\tilde{L}}_\Delta X$ wich contains F. In particular, $\theta X(F) = F$ if and only if $F \in \exp^{\tilde{L}}_\Delta X$.

We obtain a continuous retraction $\theta X : \exp(X \times \tilde{L}) \to \exp^{\tilde{L}}_\Delta X$, hence $\exp^{\tilde{L}}_\Delta X$ is a compactum.

For a closed subset $\mathcal{F} \subset \exp^{\tilde{L}}_\Delta X$, its intersection $\bigcap \mathcal{F}$ is is $\exp^{\tilde{L}}_\Delta X$, therefore is a greatest lower bound of $\mathcal{F}$. The equality

$$\bigcap \mathcal{F} = \{\ (\inf(pr_1(A)), \sup(pr_2(A))) \mid A \in \mathcal{F}^\perp\}$$

implies that $\bigcap \mathcal{F}$ is continuous w.r.t. $\mathcal{F}$. The least upper bound of $\mathcal{F}$ is equal to $\theta X(\bigcup \mathcal{F})$, hence is continuous w.r.t. $\mathcal{F}$ as well. If $\mathcal{F} \subset \exp^{\tilde{L}}_\Delta X$ is not closed, then $\sup \mathcal{F} = \theta X(\mathrm{Cl}\ (\bigcup \mathcal{F}))$. For two elements $\mathcal{F}_1, \mathcal{F}_2 \in \exp^{\tilde{L}}_\Delta X$, the join is equal to $\{\ (x, \alpha \wedge \beta) \mid (x, \alpha) \in \mathcal{F}_1, (x, \beta) \in \mathcal{F}_2\}$. The distributivity of join w.r.t. meet in $\exp^{\tilde{L}}_\Delta X$ is easily checked. Thus $\exp^{\tilde{L}}_\Delta X$ is a compact Hausdorff Lawson lattice.

We consider an operation $/ : L \times L \to L$, called division such that $\gamma/\beta = \sup\{\ \alpha \mid \alpha * \beta \le \gamma\}$ for all $(\gamma, \beta) \in L \times L$.

If $\ast : L \times L \to L$ is a lower (upper) semicontinuous operation then $/ : L \times L \to L$ is an upper (resp. lower) semicontinuous operation.

We do not have associativity of $/$, but for all $\gamma, \beta, \delta \in L$:

$$(\gamma/\beta)/\delta = \gamma/(\delta \ast \beta).$$

For all $F \subset L$ and $\alpha, \gamma \in L$ the following equalities are valid:

1) $(\inf F)/\gamma = \inf(F/\gamma)$;
2) $\alpha/(\inf F) = \sup(\alpha/F)$;
3) $(\sup F)/\gamma = \sup(F/\gamma)$;
4) $\alpha/(\sup F) = \inf(\alpha/F)$.

Let a division of elements of $\exp_\Delta^{\tilde{L}} X$ by elements of L be defined by the formula

$$C/\alpha = \{\ (x, \beta/\alpha) \mid (x, \beta) \in C\} \cup (\{\min X\} \times \tilde{L}), \quad C \in \exp_\Delta^{\tilde{L}} X, \alpha \in L.$$

It makes $\exp_\Delta^{\tilde{L}} X$ a compact Hausdorff Lawson $(L, \vee, \ast)$-semimodule.

[1] M. Akian, *Densities of idempotent measures and large deviations*, Trans. Amer. Math. Soc. **351**(11) (1999), 4515–4543

[2] O. Nykyforchyn, *Adjoints and monads related to compact lattices and compact Lawson idempotent semimodules*, Preprint, 2010

[3] J.D. Lawson, *Topological semilattices with small semilattices*, J. Lond. Math. Soc. **11** (1969) 719–724

APPROXIMATIONS OF CONTINUOUS FUNCTIONS ON FRÉCHET SPACES

Mytrofanov M.A. and Ravsky A.V.

Department of Functional Analysis, Pidstryhach Institute of Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, Lviv, 3-b Naukova Str., Ukraine

Department of Functional Analysis, Pidstryhach Institute of Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, Lviv, 3-b Naukova Str., Ukraine

E-mail address: `oravsky@mail.ru`

Using results for Banach spaces of [3] and [1], we consider approximations of a continuous function on a countable normed (real and complex) Fréchet space by analytic and $*$-analytic.

A $*$-polynomial on a linear space is a generalization of a polynomial (see [1] for details).

1. *Let X be a separable complex Fréchet space with a countable system $\{p_n\}_{n\geq 1}$ of norms and Y be a Banach space. Suppose that the space $X_n = (X, p_n)$ admits a separating $*$-polynomial for each $n \geq 1$. Let $f : X \to Y$ be a function such that there is a number $k \geq 1$ such that the sequence $\{f(x_n)\} \subset Y$ converges for each Cauchy sequence $\{x_n\}$ of X_k. Then the function f is uniformly approximable on X by $*$-analytic functions.*

2. *Let X be a separable complex Fréchet space with a countable system $\{p_n\}_{n\geq 1}$ of norms and Y be a Banach space. Suppose that the space $X_n = (X, p_n)$ admits a separating uniformly $*$-analytic function for each $n \geq 1$. Let $f : X \to Y$ be an uniformly continuous function such that there is a number $k \geq 1$ such that the function f in uniformly continuous on X_k. Then the function f is uniformly approximable on X by $*$-analytic functions.*

Also we found a criterium of the existence of an extension of a continuous function from a dense subspace of a topological space onto the space. In particular, we prove the following

1. *Let X be a Fréchet-Urysohn space, Y a regular topological space, D dense subset of X, and $f : D \to Y$ a continuous map. The map f extends to a continuous map from X to Y if and only if for each convergent in X sequence $\{x_n\}$ of D the sequence $\{f(x_n)\}$ converges.*

[1] M. Mitrofanov, *Approximation of continuous functions in complex Banach spaces* Math. notes. **86**(4) (2009), 557–570 (in Russian)

[2] R. Engelking, General topology, M.: Mir, 1986 (in Russian)

[3] J. Kurzweil,*On approximation in real Banach spaces*, Studia Math. **14** (1954), 214–231

FREE IDEMPOTENT SEMIMODULES OVER COMPACT HAUSDORFF LAWSON SEMILATTICES

O.R. Nykyforchyn

Vasyl' Stefanyk Precarpathian National University, Shevchenka 57, Ivano-Frankivsk, 76025, Ukraine

E-mail address: oleh.nyk@gmail.com

Let L be a compact Hausdorff Lawson lattice with $\alpha \oplus \beta$ and $\alpha \otimes \beta$ being resp. the join and the meet of $\alpha, \beta \in L$, a bottom element 0 and a top element 1. Let also $* : L \times L \to L$ be an operation, called *multiplication*, which is associative, infinitely distributive w.r.t. $\oplus$ in the both variables (or, equivalently, distributive in the both variables and lower semicontinuous), and the top element $1 \in L$ is a two-side unit for $*$. It implies that $*$ is isotone in the both variables, hence $\alpha * \beta \leqslant \alpha \otimes \beta$ for all $\alpha, \beta \in L$. In fact, $* = \otimes$ it the greatest of such possible operations. Another example is the unit segment $I = [0; 1]$ with the operations max, min, and the usual multiplication.

Hence $(L, \oplus, *)$ is an *idempotent semiring* [1]. If $*$ is also commutative, then $*$ is a *triangular norm* (*t-norm*) [3] on L. Nevertheless, we do not need the commutativity of $*$ in this paper.

For an idempotent semiring $\mathcal{S} = (S, \oplus, *, 0, 1)$ a (left idempotent) $\mathcal{S}$-semimodule is a set X with operations $\oplus : X \times X \to X$ and $* : S \times X \to X$ which satisfy natural conditions [1] roughly analogous to ones for vector spaces. Informally speaking, an idempotent semimodule is a vector space over an idempotent semiring. The operation $*$ is isotone in the both variables.

We call X a *compact Hausdorff Lawson* $(L, \oplus, *)$*-semimodule* if X is an $(L, \oplus, *)$-semimodule and carries a compact Hausdorff topology such that the upper semilattice $(X, \oplus)$ is a Lawson lattice and the operation $* : L \times X \to X$ is lower semicontinuous. We adopt a usual convention and often write αx instead of $\alpha * x$ for $\alpha \in L$ and $x \in X$, preserving the notation $*$ for operations $L \times L \to L$.

We denote by $\mathcal{LL}$aws the category of all compact Hausdorff Lawson lower semilattices and their continuous meet-preserving mappings. Let also $\mathcal{LL}\text{aws}_{\uparrow}$ and $\mathcal{LL}\text{aws}_{\downarrow}$ be the categories whose objects are compact

Hausdorff Lawson lower semilattices, and arrows are monotone mappings such that the preimages of all closed upper (resp. lower) sets are closed.

We denote by $(L,\oplus,*)$-$\mathcal{L}$w$\mathcal{SM}$od$_\downarrow$ the category that consists of all compact Hausdorff Lawson $(L,\oplus,*)$-semimodules and of all join-preserving (hence isotone) lower semicontinuous maps between them that are $*$-uniform, i.e. preserve multiplication by elements of L. If the operation $*: L\times L \to L$ is also upper semicontinuous (i.e. is continuous), we define two more categories. The objects of $(L,\oplus,*)$-$\mathcal{L}$w$\mathcal{SM}$od and $(L,\oplus,*)$-$\mathcal{L}$w$\mathcal{SM}$od$_\uparrow$ are compact Hausdorff Lawson $(L,\oplus,*)$-semimodules with continuous multiplication by elements of L. The morphisms in $(L,\oplus,*)$-$\mathcal{L}$w$\mathcal{SM}$od are continuous $*$-uniform mappings which preserve all suprema and infima, while the class of morphisms of $(L,\oplus,*)$-$\mathcal{L}$w$\mathcal{SM}$od$_\downarrow$ consists of all upper semicontinuous join-preserving $*$-uniform mappings between objects of this category.

Now we will construct left adjoint functors to the obvious forgetful functors $U^*: (L,\oplus,*)$-$\mathcal{L}$w$\mathcal{SM}$od $\to \mathcal{LL}$aws, $U^*_\uparrow : (L,\oplus,*)$-$\mathcal{L}w\mathcal{SM}od_\uparrow \to \mathcal{LL}aws_\uparrow$, $U^*_\downarrow : (L,\oplus,*)$-$\mathcal{L}w\mathcal{SM}od_\downarrow \to \mathcal{LL}aws_\downarrow$.

For a compact Hausdorff Lawson lower semilattice X, the product $X \times L$ is a compact Hausdorff Lawson lower semilattice as well. Let $\exp^L_\triangle X$ be the ordered by inclusion space of all closed subsets $C \subset X\times L$ such that, for all $\alpha,\beta \in L$, $x,y\in X$:

(1) $\alpha \leqslant \beta$, $x \leqslant y$, $(y,\beta)\in C$ implies $(x,\alpha)\in C$ (i.e. C is a lower subset of $X\times L$);

(2) $(x,\alpha),(x,\beta)\in C$ implies $(x,\alpha\oplus\beta)\in C$;

(3) $C \supset X\times\{0\}$.

By the closedness of C, a stronger version of (2) is valid:

(2') if $A\subset L$ and $x\in X$ are such that $(x,\alpha)\in C$ for all $\alpha\in A$, then $(x,\sup A)\in C$.

For each $F\subset X\times L$, the set

$$\theta X(F) = \{(x,\alpha)\in X\times L \mid x \leqslant \inf(\mathrm{pr}_1(F'))), \alpha \leqslant \sup(\mathrm{pr}_2(F')) \\ \text{for some } F' \subset F\cup(X\times\{0\}), F'\neq\varnothing\}$$

is a least subset of $X\times L$ that contains F and satisfies (1), (2'), (3). It becomes more obvious if one observe that

$$\theta X(F) = \{(x,\alpha)\in X\times L \mid \alpha = 0, \text{ or } \alpha\leqslant \sup A \\ \text{for some } A\subset L \text{ such that for all } \beta\in A \text{ there is } (y,\beta)\in F, x\leqslant y\}.$$

In particular, $\theta X(F) = F$ if and only if F satisfies (1), (2'), (3). Observe that the closure of a subset $C \subset X \times L$, that satisfies (1), (2'), (3), satisfies these properties as well, hence $\Theta X(F) = \mathrm{Cl}(\theta X(F)) = \theta X(\mathrm{Cl}\, F)$ is a least element of $\exp^L_\triangle X$ that contains F. It is equal to

$$\Theta X(F) = \{(x, \alpha) \in X \times L \mid \alpha = 0, \text{ or for all } \\ \alpha' \lessdot \alpha, x' \lessdot x \text{ there are } n \in \mathbb{N}, (y_1, \alpha_1), \ldots, (y_n, \alpha_n) \in F \\ \text{such that } x' \leqslant y_1, \ldots, x' \leqslant y_n, \alpha_1 \oplus \ldots \oplus \alpha_n \geqslant \alpha'\}.$$

If F is closed, then $\theta X(F)$ is closed as well, hence $\theta X(F) = \Theta X(F)$, and in this case we can equivalently take only *closed* subsets F' of $F \cup (X \times \{0\})$ in the definition. We obtain a continuous retraction $\theta X : \exp(X \times L) \to \exp^L_\triangle X$, thus $\exp^L_\triangle X$ is a compactum.

For a closed subset $\mathcal{F} \subset \exp^L_\triangle X$, its intersection $\bigcap \mathcal{F}$ is in $\exp^L_\triangle X$, therefore is a greatest lower bound of $\mathcal{F}$. The equality

$$\bigcap \mathcal{F} = \{(\inf(\mathrm{pr}_1(A)), \inf(\mathrm{pr}_2(A))) \mid A \in \mathcal{F}^\perp\}$$

implies that $\bigcap \mathcal{F}$ is continuous w.r.t. $\mathcal{F}$. The least upper bound of $\mathcal{F}$ is equal to $\theta X(\bigcup \mathcal{F})$, hence is continuous w.r.t. $\mathcal{F}$ as well. If $\mathcal{F} \subset \exp^L_\triangle X$ is not closed, then $\sup \mathcal{F} = \Theta X(\bigcup \mathcal{F})$. For two elements $\mathcal{F}_1, \mathcal{F}_2 \in \exp^L_\triangle X$, the join is equal to $\{(\alpha \oplus \beta, x) \mid (\alpha, x) \in \mathcal{F}_1, (\beta, x) \in \mathcal{F}_2\}$. The distributivity of join w.r.t. meet in $\exp^L_\triangle X$ is easily checked. Thus $\exp^L_\triangle X$ is a compact Hausdorff Lawson lattice. Its bottom and top elements are equal to $X \times \{0\}$ and $X \times L$ respectively.

Let the multiplication $* : L \times \exp^L_\triangle X \to \exp^L_\triangle X$ be defined as follows: for a set $C \in \exp^L_\triangle X$ and $\alpha \in L$, the product αC is the least element of $\exp^L_\triangle X$ that contains the set $\{(x, \alpha * \beta) \mid (x, \beta) \in C\}$, i.e.

$$\alpha C = \Theta X(\{(x, \alpha * \beta) \mid (x, \beta) \in C\}).$$

There is an embedding $\eta^L_\triangle X : X \hookrightarrow \exp^L_\triangle X$ that sends each $x \in X$ to $(X \times \{0\}) \cup (\{x\}{\downarrow} \times L)$.

1. *The semimodule* $\exp^L_\triangle X$ *together with the mapping* $\eta^L_\triangle X : X \to \exp^L_\triangle X$ *is a free object over* X *(as an object of* $\mathcal{LL}\mathrm{aws}$, $\mathcal{LL}\mathrm{aws}_\uparrow$, *and* $\mathcal{LL}\mathrm{aws}_\downarrow$*) in resp.* $(L, \oplus, *)$-$\mathcal{L}\mathrm{w}\mathcal{SM}\mathrm{od}$, $(L, \oplus, *)$-$\mathcal{L}\mathrm{w}\mathcal{SM}\mathrm{od}_\uparrow$, *and* $(L, \oplus, *)$-$\mathcal{L}\mathrm{w}\mathcal{SM}\mathrm{od}_\downarrow$.

[1] Akian, M.: Densities of invariant measures and large deviations. Trans. Amer. Math. Soc. **351(11)**, 4515–4543 (1999)
[2] Barr, M., Wells, Ch.: Toposes, Triples and Theories. Springer, N.Y., 1988
[3] Drossos, C.A.: Generalized t-norm structures. Fuzzy Sets and Systems. **104(1)**, 53-59 (1999)
[4] Kolokoltsov, V.N., Maslov, V.P.: Idempotent Analysis and Its Applications. Kluwer Acad. Publ., Dordrecht, 1998
[5] Lawson, J.D.: Topological semilattices with small semilattices. J. Lond. Math. Soc. **11**, 719–724 (1969)
[6] Mac Lane, S.: Categories for the Working Mathematician, 2nd ed. Springer, N.Y., 1998

ERGODIC PROPERTIES OF THE Q_∞–EXPANSION OF REAL NUMBERS AND THEIR APPLICATIONS IN NUMBER THEORY

R. Nikiforov and G. Torbin

Institute of Physics and Mathematics, National Pedagogical Dragomanov University, Kyiv, Pyrogova 9, Ukraine
E-mail address: rnikiforov@gmail.com

Institute of Physics and Mathematics, National Pedagogical Dragomanov University, Kyiv, Pyrogova 9, Ukraine
E-mail address: torbin7@gmail.com

Let $Q_\infty = (q_0, q_1, \dots, q_k, \dots)$ be a stochastic vector such that $q_i > 0$, and $-\sum\limits_{i=0}^{\infty} q_i \ln q_i < +\infty$. For any $x \in [0,1)$ there exists a unique sequence $\{\alpha_k(x)\}$ of non-negative integers such that

$$(1) \qquad x = \beta_1(x) + \sum_{k=2}^{\infty} \beta_k(x) \cdot \prod_{j=1}^{k-1} q_{\alpha_j(x)} =: \Delta_{\alpha_1(x)\alpha_2(x)\dots\alpha_k(x)\dots},$$

where $\beta_k(x) = \sum_{i=0}^{k-1} q_i$ with $\sum_{i=0}^{-1} q_i := 0$.

Expression (1) is said to be the polybasic Q_∞-expansion for real numbers.

Let $N_i(x,k)$ be a number of the digit "i" among the first k digits of the Q_∞-expansion of x.

If the limit $\lim\limits_{k\to\infty} \frac{N_i(x,k)}{k} =: \nu_i^{Q_\infty}(x)$ exists, then its value is said to be the asymptotic frequency of the digit "i" in the Q_∞-expansion of x.

1. *For λ-almost all $x \in [0,1)$ holds*

$$\nu_i(x) = q_i \quad (i \in \{0,1,2,\dots\}).$$

and

$$\lim_{n\to\infty} \sqrt[n]{q_{\alpha_1(x)} q_{\alpha_2(x)} \cdot \ldots \cdot q_{\alpha_n(x)}} = e^{-H}.$$

Let Φ be a covering system which consist of Q_∞-cylinders of $[0,1)$, i.e.,

(2) $\quad \Phi = \{E : E = \Delta_{\alpha_1\ldots\alpha_n},\ \ n \in N,\ \alpha_i \in N \cup 0,\ i = 1,2,...,n\},$

and let $\dim_H(E,\Phi)$ be the Hausdorff dimension of set $E \subset [0,1)$ with respect to the covering system Φ.

2. *If* $q_i = \frac{1}{2^i}$, *then* $\dim_H(E,\Phi) = \dim_H E, \forall E \subset [0,1)$.

The set

$$N(Q_\infty) = \left\{ x:\ \exists i : \nu_i^{Q_\infty}(x) \neq q_i \text{ or } \lim_{k\to\infty} \frac{N_i(x,k)}{k} \text{ does not exist } \right\}$$

is said to be the set of Q_∞-non-normal numbers.

3.

$$\dim_H(N(Q_\infty)) = 1.$$

DYNAMICAL COMPACTIFICATIONS

I.V. Protasov

Department of Cybernetics, Kyiv National University, Volodimirska 64, Kyiv 01033, Ukraine
E-mail address: i.v.protasov@gmail.com

1. Uniform compactifications

Given a set X and $U, V \subseteq X \times X$ we put,

$$UV = \{(x, y) : (x, z) \in U, (z, y) \in V \text{ for some } z \in X\}$$

$$U^{-1} = \{(y, x) : (x, y) \in U\}$$

A *uniform structure* (or *uniformity*) $\mathcal{U}$ on X is a filter of subsets of $X \times X$ with the following properties:

(1) $\Delta \subseteq U$ for every $U \in \mathcal{U}$, where $\Delta = \{(x, x) : x \in X\}$;
(2) for every $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$;
(3) for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ for which $V^2 \subseteq U$.

Let $\mathcal{U}$ be an uniformity on X and let $U \in \mathcal{U}$. For any $x \in X$ and $Y \subseteq X$, we put

$$U(x) = \{y \in X : (x, y) \in U\}, \quad U[Y] = \bigcup_{y \in Y} U(y)$$

Then $\mathcal{U}$ generates a topology on X in which a base of neighbourhoods of the point $x \in X$ are the sets of the form $U(x)$, where $U \in \mathcal{U}$. If X has this topology, $(X, \mathcal{U})$ is called a *uniform space*.

If $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces, a function $f : X \to Y$ is said to be *uniformly continuous* if, for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $(f(x_1), f(x_2)) \in V$ whenever $(x_1, x_2) \in U$.

A topological space X is called *uniformirable* if its topology can be generated by some uniformity on X. Metric spaces and topological groups provide important examples of uniformirable spaces.

If (X, d) is a metric space, the filter which has as base the sets of the form $\{(x, y) \in X : d(x, y) < r\}$, where $r > 0$, is a uniformity on X. This example includes all discrete spaces. If X is discrete, it has the trivial uniformity $\mathcal{U} = \{U \subseteq X \times X : \Delta \subseteq U\}$.

If G is a topological group, its topology is defined by the right uniformity which has as base the sets $\{(x, y) \in G \times G : xy^{-1} \in V\}$, where V denotes a neighbourhood of identity. We shall assume that we have assigned this uniformity to any topological group to which we refer.

It is precisely the completely regular topological spaces which are uniformirable. X is said to be completely regular if, for every closed subset E of X and every $x \in X \setminus E$, there is a function $f \in C_{\mathbb{R}}(X)$ for which $f(x) = 0$, $f[E] = \{1\}$. For each $f \in C_{\mathbb{R}}(X)$ and each $\varepsilon > 0$, we put $U_{f,\varepsilon} = \{(x, y) \in X \times X : |f(x) - f(y)| < \varepsilon\}$. The finite intersections of the sets of the form $U_{f,\varepsilon}$, then provide a base for a uniform structure on X.

In particular, every compact space is uniformirable. In fact, X has a unique uniform structure given by the filter of neighbourhood of the diagonal in $X \times X$.

A *topological compactification* of a space X is a pair (φ, Y), where Y is a compact space, $\varphi : X \to Y$ is a topological embedding and $\varphi[X]$ is dense in Y.

Let $(X,\mathcal{U})$ be a uniform space. There is a topological compactification $(\gamma, \gamma X)$ of X such that it is precisely the uniformly continuous functions in $C_{\mathbb{R}}(X)$ which have continuous extensions to γX. That is $\{f \in C_{\mathbb{R}}(X) : f = g \circ \varphi$ for some $g \in C_{\mathbb{R}}(\gamma X)\} = \{f \in C_{\mathbb{R}}(X) : f$ is uniformly continuous$\}$.

Since γ is an embedding, we shall regard X as being a subspace of γX. The compactification γX will be called the *uniform compactification* of X. It has the following universal property.

Let X, Y be uniform spaces, $f : X \to Y$ be a uniformly continuous mapping. Then there exists a continuous extension $f^\gamma : \gamma X \to \gamma Y$.

The construction of γX is based on the next lemma which establish a relation between compactifications of X and subalgebras of $C_{\mathbb{R}}(X)$.

Let X be any topological space and let A be a norm closed subalgebra of $C_{\mathbb{R}}(X)$ which contains the constant functions. There is a compact space Y and a continuous function $\varphi : X \to Y$ with the property that $\varphi[X]$ is dense in Y and $A = \{f \in C_{\mathbb{R}}(X) : f = g \circ \varphi$ for some $g \in C_{\mathbb{R}}(Y)\}$. The mapping φ is an embedding if, for every closed subset E of X and every $x \in X \setminus E$, there exists $f \in A$ such that $f(x) = 0$ and $f[E] = \{1\}$.

If X is discrete, γX coincides with with the Stone-Čech compactification βX of X and can be described as follows. We take the points of βX to be the ultrafilters on X, with the points of X identified with the principal ultrafilters, and denote by $X^* = \beta X \setminus X$ the set of all free ultrafilters on X. The topology of βX can be defined by stating that the sets of the form $\overline{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, are a base for the open sets.

2. Greatest G-ambit and enveloping semigroup

Let G be a topological group with the identity e. A *G-space* is a topological space X with a continuous action of G, that is, a mapping $G \times X \to X$, $(g, x) \mapsto gx$ satisfying $g(hx) = (gh)x$ and $ex = x$ for all $g, h \in G$ and $x \in X$.

A *G-mapping* is a continuous mapping $f : X \to Y$ between G-spaces such that $f(gx) = g(f(x))$ for all $x \in X, g \in G$.

A compact G-space X with a distinguished point x is called a *G-ambit* if the orbit Gx of x is dense in X.

A *morphism* between G-ambits (X, x) and (Y, y) is a G-mapping $X \to Y$ taking x to y.

Recall that a function $f : G \to \mathbb{R}$ is *right uniformly continuous* if

$$\forall \varepsilon > 0\ \exists V \in \mathcal{N}(G)\ \forall x \forall y \in G :$$

$$xy^{-1} \in V \Rightarrow |f(y) - f(x)| < \varepsilon,$$

where $\mathcal{N}(G)$ is the filter of neighbourhood of e.

We denote by $\mathcal{R}$ the right uniformity on G and by γG the uniform compactification of $(G, \mathcal{R})$. The G-space γG has a distinguished point e and the G-ambit $(\gamma G, e)$ has the following universal property:

for every compact G-space X and every $p \in X$, there exists a unique G-mapping $f : \gamma G \to X$ such that $f(e) = p$, so γG is the greatest G-ambit.

For every topological group G, the greatest G-ambit γG has a natural structure of compact right-topological semigroup with the identity e such that the multiplication $\gamma G \times \gamma G \to \gamma G$ extends the action $G \times \gamma G \to \gamma G$. Given $x, y \in X$, in virtue of the universal property of X, there is a unique G-mapping $r_y : \gamma X \to \gamma X$ such that $r_y(e) = e$, so we put $xy = r_y(x)$.

For a discrete group G, the product pq of the ultrafilters can be defined by the rule: given $A \subseteq G$,

$$A \in pq \Leftrightarrow \{g \in G : g^{-1}A \in p\} \in q.$$

To define an enveloping semigroup of G-space X, we note that the space ${}^X X$ provided with topology of point-wise convergence has a natural structure of compact right-topological semigroup (with operation of composition) in which all the left shifts $g \mapsto fg$ are continuous provided that $f \in {}^X X$ is continuous. The enveloping semigroup $\mathcal{E}(X)$ is the closure in ${}^X X$ of the set $\{g(x) : g \in G\}$. The action of G on $\mathcal{E}(X)$ is defined by $f(x) \mapsto f(gx)$, $g \in G$.

The enveloping semigroup $\mathcal{E}(X)$ of G-space X is the greatest G-ambit with the property that morphisms into X separate points. In other words, morphisms of G-space $\mathcal{E}(X) \to X$ separate points in $\mathcal{E}(X)$, and whenever (Z, z) is a G-ambit such that morphisms of G-spaces $Z \to X$ separate points in Z, there is unique morphism of G-spaces $(\mathcal{E}(X), id_X) \to (Z, z)$.

Let G be a discrete group. The shift system over G is topologically Cantor cube ${}^G\mathbb{Z}_2$, upon which G acts by left translations. The enveloping semigroup $\mathcal{E}({}^G\mathbb{Z}_2)$ is isomorphic to the greatest G-ambit γG.

3. Universal minimal G-spaces and extremal amenability

A G-space is *minimal* if it has no proper G-invariant closed subset or, equivalently, if the orbit Gx is dense in X for every $x \in X$. The *universal minimal compact* G-space μG is characterized by the following property: μG is minimal and, for every minimal compact G-space X there exists a G-mapping of μG onto X.

For every topological group G, there exists universal minimal compact G-space μG, which is unique up to G-isomorphisms. Every minimal closed left ideal L of the greatest ambit γG is a minimal compact G-space, moreover, L is a retract of γG.

In some cases, the space μG can be described explicitly. For example, let E be a countable infinite discrete space, and let $G = Sym(E) \subset {}^E E$ be the topological group of all permutations of G. Then μG can be identified with the space of all linear orders on E. Every linear order is considered as a subset of $E \times E$ is identified with the compact space ${}^{E\times E}\{0, 1\}$.

Another example is the following. Let S^1 be a circle, and let $G = H_+(S^1)$ be the group of all orientation-preserving homeomorphisms of S^1. Then μG can be identified with S^1. If K is a compact manifold of dimension > 1 and $H(X)$ is the group of homeomorphisms of K, then $\mu G \neq K$ in view of the following general result:

For every topological group G, the action of G on the minimal compact G-space μG is not 3-transitive.

A topological group G is called *extremally amenable* if every compact G-space X has a G-fixed point x, i.e. $gx = x$ for every $g \in G$. Equivalently, G is extremally amenable if μG is a singleton.

Recall that a topological group G is *amenable* if every continuous action of G by affine transformations on a convex compact subset of a locally convex vector space has a G-fixed point.

A subset A of a group G is called *left large* if there exists a finite subset F of G such that $G = FA$.

A topological group G is extremally amenable if and only if whenever $A \subseteq G$ is left large, AA^{-1} is dense in G.

Let us say that a group G of order-preserving automorphisms of a linearly ordered set X is ω-transitive if it takes any finite subset to any finite subset of the same size.

An ω-transitive group of order automorphisms of an infinite linearly ordered set X, equipped with the topology of point-wise convergence on X, is extremally amenable. The group $Aut(Q, \leqslant)$, considered as a discrete group has a common fixed point on each compact metric space.

A necessary condition for a group G to be extremaly amenable is that there be no non-constant continuous characters $\chi : G \to \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Indeed, if $\chi : G \to \mathbb{T}$ is a character, $\chi \neq 1$, then G admits a fixed-point free action on $\mathbb{T}$ given by $(g, x) \mapsto \chi(g)x$.

Let G be an Abelian topological group. Suppose that G has no non-trivial characters $\chi : G \to \mathbb{T}$. Is G extremaly disconnected?

For cyclic group the question can be reformulated as follows. Let K be a compact space, and let $f \in H(K)$ be a fixed-point free homeomorphism of K. Let C be the cyclic subgroup of $H(K)$ generated by f. Does there exist a complex number a such that $|a| = 1$, $a \neq 1$, and the homeomorphism $\chi : G \to \mathbb{T}$ defined by $\chi(f^n) = a^n$ is continuous?

In the case $G = \mathbb{Z}$, a negative answer to this question would imply a negative answer to the following long-standing problem. We remind that a Bohr topology on $\mathbb{Z}$ is the strongest precompact group topology.

Let A be a large subset of $\mathbb{Z}$. Is the set $A - A$ a Bohr neighbourhood of zero in $\mathbb{Z}$?

The above question has also a purely combinatorial equivalent.

Let A be a large subset of $\mathbb{Z}$. Does there exist a large subset B such that $B - B + B - B \subseteq A - A$?

For a topological group G the following statements are equivalent

(i) the canonical morphism $\gamma G \to \mathcal{E}(\mu G)$ is an isomorphism;
(ii) points of γG are separated by G-mappings to the minimal G-spaces.

For precompact group G, (ii) holds because $\gamma G = \mu G$. For the group $\mathbb{Z}$ with the discrete topology, (ii) does not hold.

Is a topological group precompact provided that the points of γG are separated by G-mappings to the minimal G-spaces?

4. Dynamical equivalences and coronas

Let G be a topological group, X be a G-space. The *orbit equivalence* E on X ($(x,y) \in E \Leftrightarrow \exists g \in G : gx = y$) produces the following three derived equivalences on X

($\mathring{E}$): $(x,y) \in \mathring{E} \Leftrightarrow clE_x = clE_y$, where E_x, E_y are E-equivalence classes containing x and y;

($\dot{E}$): $\dot{E}$ is the smallest by inclusion equivalence on X containing E such that every $\dot{E}$-equivalence class is closed;

($\check{E}$): $\check{E}$ is the smallest by inclusion closed in $X \times X$ equivalence on X containing E.

For every infinite discrete group G, the remainder $G^* = \beta G \setminus G$ of the Stone-Čech compactification βG of G has a natural structure of G-space. We describe the interrelations between the classes of the equivalences $\mathring{E}$, $\dot{E}$, $\check{E}$ and the principal left ideals of the semigroup βG.

The factor-space $\nu G = G^*/\check{E}$ is called a corona of G and can be considered as a topological orbit space of G^*. To clearify the virtual equivalence $\check{E}$ we use the slowly oscillating functions. A function $f : G \to [0,1]$ is called *slowly oscillating* if, for all $\varepsilon > 0$ and $g \in G$, there exists a finite subset F of G such that $|f(x) - f(gx)| < \varepsilon$ for every $g \in G \setminus F$.

Given any $p, q \in G^*$, we have $(p,q) \in \check{E}$ if and only if, for every slowly oscillating function $f : G \to [0,1]$, $f^\beta(p) = f^\beta(q)$.

For every countable discrete group G, νG contains a topological copy of $\omega^* = \beta\omega \setminus \omega$ and there exists a continuous surjective mapping $f : \nu G \to \nu\mathbb{N}$, where $\nu\mathbb{N} = \{\check{p} \in \nu\mathbb{Z} : \mathbb{N} \in p\}$. Moreover, if G is locally finite, then νG contains a topological copy of ω^* which is a retract of νG.

Besides the equivalences $\mathring{E}$, $\dot{E}$, $\check{E}$ on G^*, we consider also the tent relation $\hat{E}$ defined by

$$(x,y) \in \hat{E} \Leftrightarrow clE_x \subseteq clE_z, clE_y \subseteq clE_z \text{ for some } z \in G^*,$$

which is also an equivalence if G is countable. Then we have

$$E \subset \mathring{E} \subset \hat{E} \subseteq \dot{E} \subset \check{E}.$$

Is $\hat{E} = \dot{E}$ for the orbit equivalence E on $\mathbb{Z}^*$?

[1] N.Hindman and D.Strauss, *Algebra in the Stone-Čech compactification - Theory and Applications*, de Grueter, Berlin, 1998.

[2] V.Pestov, *Some universal constructions in abstract topological dynamics*, Contemporary Mathematics, **215**(1998), 83-99.

[3] V.Pestov, *Forty annotated questions about large topological groups*, arXiv:math.GN/0512564 v1 26 Dec 2005

[4] I.V.Protasov, *Coronas of balleans,* Topology Appl., **149**(2005), 149-160.

[5] I.V.Protasov, *Dynamical equivalences on G^**, Topology Appl., **155**(2008), 1394-1402

[6] V.Uspenskij, *Compactifications of topological groups*, Proceedings of the Ninth Prague Topological Symposium, (Prague 2001), 331-346, Topology Atlas, Toronto, 2002, arXiv:math.GN/0204144

ON ONE HYPERSPACE OF SUBSETS OF THE HILBERT CUBE

Yu. Shatskiy

Mathematics and informatics department, Precarpathian National University named after V. Stefanyk, Ivano-Frankivsk, 57 Shevchenko str., 76025, Ukraine

E-mail address: `yurashac@gmail.com`

We consider the hyperspace $M_\varepsilon(Q)$ that consists of the closed subsets of the Hilbert cube such that all their points are in segments of fixed length $\varepsilon > 0$ which are entirely contained within the mentioned subsets. Its topological and geometrical properties are studied. In particular it is proved that $M_\varepsilon(Q)$ is a metric compactum, a Lawson compact topological upper semilattice [2], and, under additional assumptions about ε, an absolute retract.

[1] В.В. Федорчук, В.В. Филиппов. Общая топология: основные конструкции. М.: Наука, 1989.

[2] J.D. Lawson, *Topological semilattices with small semilattices,* J. Lond. Math. Soc. 11 (1969) 719-724.

NATURAL TRANSFORMATION OF FUNCTORS IN THE ASYMPTOTIC CATEGORY

Oksana Shukel'

Department of Geometry and Topology, Ivan Franko Lviv National University, Lviv, Universytetska Street 1, Ukraine

E-mail address: oshukel@gmail.com

The objects of the asymptotic category are proper metric spaces and the morphisms are proper asymptotically Lipschitz maps [1] .

To our purposes, it is reasonable to modify the asymptotic category and to assume that its objects are discrete metric spaces. Then the morphisms are the Lipschitz maps. In [2], the author introduced the construction that assigns to every normal functor in the category of compact Hausdorf spaces in the sense of E. Shchepin [3] a functor F in the asymptotic category.

For every proper metric space (X, d) a metric $\hat{d}$ on the space $F(X)$ is defined as follows.

Given $a, b \in F(X)$, we let

$$\hat{d}(a,b) = \inf\{\sum_{i=1}^{m} d(f_{2i-1}, f_{2i}) \mid f_{2i-1}, f_{2i}\colon A_i \to X \text{ are such that}$$

$$\text{there exist } \ c_i \in F(A_i),\ \operatorname{supp}(c_i) = A_i,\ i = 1, \dots, m, \text{ with}$$

$$a = F(f_1)(c_1),\ F(f_2)(c_1) = F(f_3)(c_2), \dots,$$

$$F(f_{2m-1})(c_m) = F(f_{2m-2})(c_{m-1}),\ F(f_{2m})(c_m) = b\}.$$

The aim of this talk is to extend the mentioned construction onto the case of natural transformation of finite normal functors.

Recall that any natural transformation $\phi\colon F \to G$ consists of a collection of morphisms $(\phi_X\colon F(X) \to G(X))_X$ such that for every $f\colon X \to Y$ we have $\phi_Y F(f) = G(f)\phi_X$.

Теорема 1. *Any natural transformation of finite normal functors of finite degree generates a (unique) natural transformation of the corresponding functors in the asymptotic category.*

Since the Hausdorff metric d_H on the hypersymmetric powers $\exp_n X$ is equivalent to the metric $\hat{d}$ defined by means of the mentioned construction [2], one can define the natural transformation of support $\mathrm{supp}\colon F \to \exp_n$.

As an application, one can extend the class of functors in the asymptotic topology.

Теорема 2. *Let $\phi\colon F \to G$ be a natural transformation of finite normal functors of finite degree and let $H \subset G$ be a subfunctor. Then $\phi^{-1}(H)$ is a normal functor of finite degree.*

Теорема 3. *Let $H \subset F$ be a subfunctor of a finite normal functor of finite degree and let $\phi\colon H \to G$ be a natural transformation, where G is also a finite normal functor of finite degree. Define $(F \cup_\phi G)(X) = (F(X) \sqcup G(X))/\sim$, where $a \sim \phi_X(a)$, for every $a \in H(X)$. Then $F \cup_\phi G$ is a normal functor of finite degree.*

The latter theorem describes the gluing operation for functors.

Література

[1] A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys. **55** (6) (2000), 71–116.

[2] O. Shukel', *Functors of finite degree and asymptotic dimension zero*, Matematychni Studii. **29** (1) (2008), 101–107.

[3] E.V. Shchepin, *Functors and uncountable powers of compacta*, Uspekhi Mat. Nauk. **36** (3) (1981), 3–62.

ON NODAL ALGEBRAS

D.E. Voloshyn

Institute of Mathematics of NAS of Ukraine, Kyiv
E-mail address: denys_vol@ukr.net

The class of nodal algebras first was considered in [1], where it was shown that nodal algebras are unique pure noetherian algebras such that the classification of their modules of finite length is tame (all others being wild).

Definition. A noetherian ring is called *pure noetherian* if it has no minimal ideals. A ring N is called *nodal* if it is semi-perfect and pure noetherian, and there is a hereditary [2, 3] ring $H \supseteq N$, which is also semi-perfect and pure noetherian such that

1) $\operatorname{rad} N = \operatorname{rad} H$;
2) $\operatorname{length}_N(H \otimes_N U) \leqslant 2$ for every simple left N-module U and $\operatorname{length}_N(V \otimes_N H) \leqslant 2$ for every simple right N-module V.

We describe nodal algebras over $\mathbb{K}[[t]]$ where $\mathbb{K}$ is an algebraically closed field. This characterization can be used to describe vector bundles over certain noncommutative projective curves [4].

[1] Y.A. Drozd, *Finite modules over pure Noetherian algebras*, Trudy Mat. Inst. Steklov Acad. Nauk USSR **183** (1990), 56–68 (English translation: Proc. Steklov Inst. of Math. **183** (1991), 97–108)

[2] C. Faith, Algebra: Rings, Modules and Categories I, Springer-Verlag, New York, 1973 (Russian translation: Mir, Moscow, 1977)

[3] G. Michler, *Structure of semi-perfect hereditary rings*, J. Algebra **13**(3) (1969), 327–344

[4] D.E. Voloshyn, *Vector bundles over noncommutative nodal curves*, 7^{th} Intern. Algeb. Conf. in Ukraine, Kharkiv 2009, 148

ALGEBRAS OF ANALYTIC FUNCTIONS IN BANACH SPACES

A.V. Zagorodnyuk

Faculty of Mathematics and Informatics, Vasyl Sefanyk Precarpathian National University, 57 Shevchenka Str., Ivano-Frankivsk 76000, Ukraine
E-mail address: andriyzag@yahoo.com

Introduction

Let A be a complex commutative topological algebra. Let us denote by $M(A)$ the spectrum (set of continuous characters = set of continuous complex-valued homomorphisms) of A. It is well known from the Theory of commutative algebras that there is a bijective correspondents between maximal ideals of A and its complex homomorphisms. So, we will identify $M(A)$ with the set of maximal ideals of A.

Recall that A is a *semisimple* algebra if the complex homomorphisms in $M(A)$ separate points of A. It is well known that every semisimple commutative Fréchet algebra A is isomorphic to some subalgebra of continuous functions on $M(A)$ endowed with a natural topology. More exactly, for every $a \in A$ there exists a function $\widehat{a} : M(A) \to \mathbb{C}$ defined by $\widehat{a}(\phi) := \phi(a)$. The weakest topology on $M(A)$ such that all functions $\widehat{a}$, $a \in A$, are continuous is called the *Gelfand topology*. The Gelfand topology coincides with the weak-star topology of the strong dual space A', restricted to $M(A)$. If A is a Banach algebra, $M(A)$ is a weak-star compact subset of the unit ball of A'.

The map

$$A \ni a \rightsquigarrow \widehat{a} \in C(M(A))$$

is called the *Gelfand transform* of A, where $C(M(A))$ is the algebra of all continuous functions on $M(A)$.

If A is a uniform algebra of continuous functions on a metric space G, then for any $x \in G$ the *point evaluation functional* $\delta(x) : f \mapsto f(x)$ belongs to $M(A)$.

Let us consider several important examples. Let G be a metric spaces and $C_b(G)$ be the uniform Banach algebra of all bounded continuous functions on G. Then the spectrum of $C_b(G)$ coincides with the the Czech-Stone compactification βG of G. That is, every function $f \in C_b(G)$ can be extended to a continuous function $\widehat{f}$ on βG and for every point $\xi \in \beta G$ the map $f \mapsto \widehat{f}(\xi)$ is a complex homomorphism of $C_b(G)$.

Let $A(\Omega)$ be a uniform algebra of all analytic functions on an open domain $\Omega \in \mathbb{C}^n$ which are continuous on the closure $\overline{\Omega}$. Then $M(A(\Omega))$ is the *polynomial convex hull* $[\Omega]$ of Ω (see [24] for details), where $[\Omega]$

is defined as a subset of $\mathbb{C}^n$ such that for every polynomial f, $|f(x)| \leq \sup_{z\in\Omega} |f(z)|$. A set is *polynomially convex* if it coincides with its polynomial convex hull. If Ω is convex, then $[\Omega] = \overline{\Omega}$. In particular, if $\Omega = \mathbb{C}^n$, then $A(\Omega)$ is the algebra of all entire functions on $\mathbb{C}^n$, $H(\mathbb{C}^n)$ and its spectrum coincides with the point evaluation functionals of $\mathbb{C}^n$.

Following these examples we can think the spectrum of an uniform algebra as a maximal natural domain such that all elements of this algebra can be considered as a continuous function on this domain. Our purpose is investigation of the spectra of various algebras of analytic functions.

1. Algebras of Polynomials

1.1. **Introduction to Polynomials.** Let X and Y be complex Banach spaces. For every positive integer numbers $n, m \in \mathbb{N}$, X^nY^m will denote the Cartesian product of n copies of X and m copies of Y and x^ny^m will denote the element $(x, \dots, x, y, \dots, y)$ from X^nY^m.

For $n \in \mathbb{N}$ we let $\mathcal{L}(^nX, Y)$ denote the space of all continuous n-linear mappings from X to Y. Let us denote by Δ_n the natural embeddings called *diagonal mappings* from X to X^n defined as

$$\begin{aligned} \Delta_n : \quad & X \to X^n \\ & x \mapsto (x, \dots, x). \end{aligned}$$

Definition 7. *A mapping P from X to Y is called a continuous n-homogeneous polynomial if $P(x) = B(\Delta_n(x))$ for some $B \in \mathcal{L}(^nX, Y)$. We let denote $\mathcal{P}(^nX, Y)$ the vector space of all continuous n-homogeneous polynomials. An n-linear mapping B is called symmetric if $B(x_1, \dots, x_n) = B(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any permutation σ on the set $\{1, \dots, n\}$. The space of all continuous symmetric n-linear maps will be denoted by $\mathcal{L}_s(^nX, Y)$.*

1. *The spaces $\mathcal{L}(^nX, Y)$ and $\mathcal{L}_s(^nX, Y)$ are Banach spaces with norms of uniform convergence on the unit ball of X^n.*

1. *The map*

$$\begin{aligned} \mathcal{L}_s(^nX, Y) \quad & \to \mathcal{P}(^nX, Y) \\ B \quad & \mapsto B \circ \Delta_n \end{aligned}$$

is an isomorphism between the Banach space $\mathcal{L}_s(^nX, Y)$ and the space $\mathcal{P}(^nX, Y)$ with norm of uniform convergence on the unit ball of X and

$$\|B \circ \Delta_n\| \leq \|B\| \leq \frac{n^n}{n!} \|B \circ \Delta_n\|. \tag{1}$$

Доведення. The main tool of the proof is *the polarization formula* (see [19, p. 8]):

$$F(x_1,\dots,x_n)=\frac{1}{2^n n!}\sum_{\epsilon_i=\pm1}\epsilon_1\dots\epsilon_n F\circ\Delta_n\Big(\sum_{j=1}^{n}\epsilon_i x_j\Big). \tag{2}$$

By the polarization formula

$$\|B\|\le\frac{1}{2^n n!}\sum_{1\le i\le n}\sum_{\epsilon_i=\pm1}\sup_{\|x_i\|\le1}\Big\|B\circ\Delta_n\Big(\sum_{j=1}^{n}\epsilon_i x_i\Big)\Big\|=$$

$$\frac{n^n}{2^n n!}\sum_{1\le i\le n}\sum_{\epsilon_i=\pm1}\sup_{\|x_i\|\le1}\Big\|B\circ\Delta_n\Big(\frac{1}{n}\sum_{j=1}^{n}\epsilon_i x_j\Big)\Big\|\le\frac{n^n}{n!}\|B\circ\Delta_n\|.$$

The left-hand side of inequality (1) is trivial. $\square$

For a positive integer n and a Banach space X let

$$c(n,X):=\inf\{M>0\colon\ \|B\|\le M\|B\circ\Delta_n\|,\ \text{for all } B\in\mathcal{L}_s(^nX,Y)\}. \tag{3}$$

We call $c(n,X)$ the nth *polarization constant* of X. From (1) it follows that

$$1\le c(n,X)\le\frac{n^n}{n!}. \tag{4}$$

It is well known that $c(n,\ell_1)=n^n/n!$ and $c(n,\ell_2)=1$ (see [20, p. 45]) for details.

2. *$\mathcal{P}(^nX,Y)$ is a Banach space and for any $P\in\mathcal{P}(^nX,Y)$ there is a unique n-linear symmetric map $A_P\in\mathcal{L}_s(^nX,Y)$, the so-called* **associated with P n-linear map,** *such that $P=A_P\circ\Delta_n$.*

Let us say that a class $\mathcal{F}(X,Y)$ of some nonlinear mappings from X to Y *admits a linearization* if there is a linear space $W(X)$ and an injective map $\mathcal{U}_{\mathcal{F}(X,Y)}:X\to W(X)$ such that for any $F\in\mathcal{F}(X,Y)$ there is a linear operator $L_F\in\mathcal{L}(W(X),Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & Y\\ \mathcal{U}_{\mathcal{F}(X,Y)}\downarrow & \nearrow & L_F\\ W(X) & & \end{array} \tag{5}$$

commutes. The map $\mathcal{U}_{\mathcal{F}(X,Y)}$ is called *the canonical map* associated with the linearization.

3. *The space $\mathcal{L}(^nX,Y)$ admits a linearization.*

Доведення. Let us denote by $X^{(n)}$ the space of all formal finite sums $\sum_i \lambda_i(x_1,\dots,x_n)$, where $\lambda_i \in \mathbb{K}$. Let I denote the subspace of $X^{(n)}$ generated by the elements of the form

$$(x_1,\dots,x_k+x_k',\dots,x_n)-(x_1,\dots,x_k,\dots,x_n)-(x_1,\dots,x_k',\dots,x_n),$$

$$(x_1,\dots,\lambda x_k,\dots,x_n)-\lambda(x_1,\dots,x_k,\dots,x_n), \qquad 1\le k\le n.$$

We let define the n-fold *tensor product* $\otimes^n X$ of X with itself by $X^{(n)}/I$. Put $x_1\otimes\cdots\otimes x_n := (x_1,\dots,x_n)+I$ and denote by i_n the n-linear mapping from X^n into $\otimes^n X$ such that $i_n:(x_1,\dots,x_n)\mapsto x_1\otimes\cdots\otimes x_n$. Then for any $B\in\mathcal{L}(^nX,Y)$ let

$$i_n^*(B)\Big(\sum_i \lambda_i x_{i1}\otimes\cdots\otimes x_{in}\Big) := \sum_i \lambda_i B(x_{i1},\dots,x_{in}).$$

The map i_n^* is well defined and $i_n^*(B)(x_{i1}\otimes\cdots\otimes x_{in}) = B(x_{i1},\dots,x_{in})$. So if $F=B$ and $\mathcal{F}(X,Y)=\mathcal{L}(^nX,Y)$, then $L_B=i_n^*(B)$, $U_{\mathcal{L}(^nX,Y)}=i_n$ and $W(X)=\otimes^n X$ in (5). $\square$

2. *The space $\mathcal{L}(^nX,Y)$ is isometrically isomorphic to the space of linear continuous operators $\mathcal{L}(\otimes_\pi^n X,Y)$ from the* **projective tensor product** *$\otimes_\pi^n X$, where $\otimes_\pi^n X$ is the completion of $\otimes^n X$ by the norm*

$$\|w\| = \inf\Big\{\sum_{i1,\dots,in}\|x_{i1}\|\dots\|x_{in}\| : w=\sum_{i1,\dots,in} x_{i1}\otimes\cdots\otimes x_{in}\Big\}.$$

Let us define the symmetric projective tensor product $\otimes_{s,\pi}^n X$ of X to itself as the closed subspace of $\otimes_\pi^n X$ generated by the vectors

$$x_1\otimes_s\cdots\otimes_s x_n := \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)},$$

where $x_i\in X$ and $\mathcal{S}_n$ is the group of permutations of the set $\{1,\dots,n\}$.

4. *The space $\otimes_{s,\pi}^n X$ is complemented in $\otimes_\pi^n X$ and the map*

$$\nu_n\Big(\sum x_{i1}\otimes\cdots\otimes x_{in}\Big) = \sum x_{i1}\otimes_s\cdots\otimes_s x_{in}$$

is a projection.

5. $\mathcal{L}(\otimes_{s,\pi}^n X,Y)\simeq\mathcal{L}_s(^nX,Y)$.

From the polarization formula and Corollary 5 it follows that

$$(6)\quad x_1\otimes_s\cdots\otimes_s x_n = \frac{1}{2^n}\sum_{1\le i\le n}\sum_{\epsilon_i=\pm 1}\epsilon_1\dots\epsilon_n\Big(\sum_{j=1}^n \epsilon_i x_i\Big)\otimes\cdots\otimes\Big(\sum_{j=1}^n \epsilon_i x_i\Big).$$

Therefore for each vector $w\in\otimes^n_{s,\pi}X$ there is a representation

$$w=\sum_{i=1}^\infty x_i^{\otimes n}=\sum_{i=1}^\infty \overbrace{x_i\otimes\cdots\otimes x_i}^{n \text{ times}}.$$

Put

$$(7)\qquad |||w||| := \inf\Big\{\sum_{i=1}^\infty \|x_i\|^n \,:\, w=\sum_{i=1}^\infty x_i^{\otimes n}\Big\}.$$

Then for any $B\in\mathcal{L}_s({}^nX,Y)$

$$|||B||| = \sup_{|||w|||=1}\|i_n^*(B)(w)\| = \|B\circ\Delta_n\|.$$

Thus we have proved the following theorem.

3. *There is an equivalent norm $|||\cdot|||$ on $\otimes^n_{s,\pi}X$ such that the space $\mathcal{L}((\otimes^n_{s,\pi}X, |||\cdot|||),Y)$ is isometric to $\mathcal{P}({}^nX,Y)$ for every Banach space Y.*

From the polarization inequality (4) and formula (7) we have the next polarization inequality for tensor products:

$$(8)\qquad \|w\|\le |||w|||\le c(n,X)\|w\|.$$

A map $P:X\to Y$ is said to be a polynomial of degree n if $P=P_0+P_1+\cdots+P_n$, where $P_0\in Y$, $P_k\in\mathcal{P}({}^kX,Y)$ and $P_n\ne 0$. The space of all polynomials from X to Y will be denoted by $\mathcal{P}(X,Y)$. We denote the spaces $\mathcal{P}({}^kX,\mathbb{C})$ and $\mathcal{P}(X,\mathbb{C})$ by $\mathcal{P}({}^kX)$ and $\mathcal{P}(X)$ respectively. Note that $\mathcal{P}(X)$ is a topological algebra with the locally convex topology of uniform convergence on bounded sets. We will use notations $\mathcal{P}({}^{\le n}X,Y)$ and $\mathcal{P}({}^{\le n}X)$ for spaces of Y-valued and $\mathbb{C}$-valued respectively, m-degree polynomials on X, $m\le n$.

$P\in\mathcal{P}(X)$ is called a *polynomial of finite type* if it is a finite sum of finite products of linear functionals. More general, if $P\in\mathcal{P}(X,Y)$, then P is a polynomial of finite type if for every linear functional $h\in Y'$, $h\circ P$ is a polynomial of finite type. The space of n-homogeneous polynomials of finite type is denoted by $\mathcal{P}_f({}^nX,Y)$. The closure of $\mathcal{P}_f({}^nX,Y)$ in the topology of uniform convergence on bounded sets is called the space of *approximable polynomials* and denoted by $\mathcal{P}_A({}^nX, Y)$. Each approximable polynomial is weakly continuous on bounded sets. The following theorem is proved in [8] by Aron and Prolla.

4. *X' has the approximation property if and only if for every n $\mathcal{P}_f({}^nX, Y)$ coincides with the space of all n-homogeneous weakly continuous polynomials for an arbitrary Banach space Y, $\mathcal{P}_w({}^nX, Y)$*

It is unknown does equality $\mathcal{P}_f({}^nX) = \mathcal{P}_w({}^nX)$ implies the approximation property of X' however, Aron, Cole and Gamelin in [5] show that if X is a reflexive Banach space without the approximation property, then $\mathcal{P}_f({}^2X \oplus X') \neq \mathcal{P}_w({}^2X \oplus X')$.

1.2. **The Aron-Berner Extension.** A given continuous n-linear mapping $B : X \times \cdots \times X \to \mathbb{C}$, B can be extended to a continuous, n-linear mapping $\widetilde{B} : X'' \times \cdots \times X'' \to \mathbb{C}$ by

$$\widetilde{B}(x_1'', \ldots, x_n'') = \lim_{\alpha_1} \ldots \lim_{\alpha_n} B(x_{\alpha_1}, \ldots, x_{\alpha_n}), \tag{9}$$

where for each k, (x_{α_k}) is a net in X converging weak-star to x_k''.

Let $P \in \mathcal{P}({}^nX)$ and B be the n-linear map associated with P. Then the *Aron-Berner extension* $\widetilde{P}$ of P is defined as $\widetilde{P} := \widetilde{B}(x, \ldots, x)$.

5. *Let $\{x_\alpha\}$ be a net in the unit ball of X that converges weak-star to z, $\|z\| < 1$. Then there is a net $\{y_\beta\}$ in the unit ball of X such that each y_β is an arithmetic mean of a finite number of x_α's, and $P(y_\beta) \to \widetilde{P}(z)$ for every polynomial P on X.*

Let I be a set of indices and $(X_i)_{i\in I}$ be a family of Banach spaces. Denote by $\ell_\infty(X_i; I)$ the ℓ_∞ direct sum of X_i's, that is, the collection of all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $(\|x_i\|)_{i\in I}$ is bounded. Then

$$\|(x_i)_{i\in I}\|_\infty := \sup_{i\in I} \|x_i\|.$$

Let $\mathcal{U}$ be an ultrafilter on I and $(x_i)_{i\in I} \in \ell_\infty(X_i; I)$. The boundedness of the map $I \to \mathbb{R}$: $I \mapsto \|x_i\|$ ensures that $\lim_{\mathcal{U}} \|x_i\|$ exists in $\mathbb{R}$. Evidently,

$$N_{\mathcal{U}} := \{(x_i) \in \ell_\infty(X_i; I) : \lim_{\mathcal{U}} \|x_i\| = 0\}$$

is a closed linear subspace of $\ell_\infty(X_i; I)$. Let us define the *ultraproduct* of the family $(X_i)_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$ as the quotient space $\ell_\infty(X_i; I)/N_{\mathcal{U}}$ equipped with the usual quotient norm. We shall denote it by $\left(\prod X_i\right)_{\mathcal{U}}$. If $X_i = X$ for each $i \in I$, we shall write $X^{\mathcal{U}}$ instead of

$\left(\prod X_i\right)_{\mathcal{U}}$ and we shall refer to $X^{\mathcal{U}}$ as the *ultrapower* of X with respect to the ultrafilter $\mathcal{U}$. The ultrapower $X^{\mathcal{U}}$ consists of elements $(x_i)_{\mathcal{U}}$, where $x_i \in X$ for every $i \in I$ and $(x_i)_{\mathcal{U}} = (y_i)_{\mathcal{U}}$ if $\lim_{\mathcal{U}} x_i = \lim_{\mathcal{U}} y_i$.

The ultrafilter $\mathcal{U}$ on X associated with the weak convergence is called a *local ultrafilter for* X.

There are two approaches for construction of extensions of polynomials from a Banach space to its ultrapower. Let $P \in \mathcal{P}(^n X)$ and B_P be the symmetric n-linear functions associated with P. Then we define an n-linear functions on $X^{\mathcal{U}}$ by

$$\widetilde{B}_P(x_1, \ldots, x_n) = \lim_{i_1, \mathcal{U}} \ldots \lim_{i_n, \mathcal{U}} \widetilde{B}_P(x_1^{(1)}, \ldots, x_n^{(n)})$$

for $x_k = (x_i^{(k)})_{\mathcal{U}}$. It is easy to see that $\widetilde{B}_P$ is well defined, $\widetilde{B}_P$ is an extension of $\widetilde{B}_P$ and that $\|\widetilde{B}_P\| = \|B_P\|$. Thus we can define an extension of P to $X^{\mathcal{U}}$ by

$$\widetilde{P}((x_i)_{\mathcal{U}}) = \widetilde{B}_P((x_i)_{\mathcal{U}}, \ldots, (x_i)_{\mathcal{U}}).$$

If $\mathcal{U}$ is the local ultrafilter on X then the restriction of $\widetilde{P}$ to the canonical image of X'' in $X^{\mathcal{U}}$ coincides with the Aron-Berner extension of P to X''. Note that if B_P is symmetric, it does not necessary follow that $\widetilde{B}_P$ is symmetric.

6. *The following assertions are equivalent:*

(1) *For every ultrafilter $\mathcal{U}$ and every continuous symmetric bilinear function B on X, the ultrapower extension $\widetilde{B}_P$ is symmetric.*

(2) *For every ultrafilter $\mathcal{U}$ and every continuous symmetric n-linear function B on X, the ultrapower extension $\widetilde{B}_P$ is symmetric.*

(3) *For local ultrafilter on X and every continuous symmetric bilinear function B on X, the ultrapower extension $\widetilde{B}_P$ from X into X'' is symmetric.*

(4) *Every continuous symmetric linear operator from X into X' is weakly compact.*

(5) *Every continuous symmetric bilinear function on X extends to a separately weak-star continuous bilinear function on X''.*

A Banach space X is said to be *symmetrically regular* if the assertions 1-5 of Theorem 6 holds.

Since every polynomial $P \in \mathcal{P}(^nX)$ is bounded on bounded nets, we can define

$$\overline{P}((x_i)_{\mathcal{U}}) := \lim_{\mathcal{U}} P(x_i)$$

and we have $\|P\| = \|\overline{P}\|$. Note that, in general, $\widetilde{P} \neq \overline{P}$.

A closed subspace Y of a Banach space X is *locally complemented* in X if there is a constant M such that whenever F is a finite-dimensional subspace of X there is a linear map (depending on the given finite-dimensional subspace) $T : F \to X$ so that $\|T\| \leq M$ and $Tx = x$ for all $x \in F \cap X$.

Thus, for instance, the Principle of Local Reflexivity of Lindenstrauss and Rosenthal says that every Banach space is locally complemented in its bidual. Also, it is well-known that every Banach space is locally complemented in its ultrapowers

7. *Let Y be a subspace of X. Then there exists a linear extension operator $\mathcal{P}(^nY) \to \mathcal{P}(^nX)$ for all (or some) $n \geq 1$ if and only if Y is locally complemented in X.*

1.3. **Spectra of Algebras of Polynomials.**

6. (Aron, Cole, Gamelin). *Let Y be a complex vector space. Let $F = (f_1, \ldots, f_n)$ be a map from Y to $\mathbb{C}^n$ such that the restriction of each f_j to any finite dimensional space of Y is a polynomial. Then the closure of the range of F is an algebraic variety.*

Доведення. Let Y_0 be a finite dimensional subspace of Y. It is well known to algebraic geometry that the closure $F(Y_0)^-$ of $F(Y_0)$ is an irreducible algebraic variety of dimension $k \leq n$. Without loss of generality, we can assume that Y_0 is chosen so that the dimension k of $F(Y_0)^-$ is a maximum. If Y_1 is any finite dimensional subspace of Y such that $Y_1 \supseteq Y_0$ then $F(Y_1)^-$ is also an irreducible algebraic variety of dimension k, which contains $F(Y_0)^-$. It follows that $F(Y_1)^- = F(Y_0)^-$, and we conclude that $F(Y_0)^- = F(Y)^-$. $\square$

8. (Aron, Cole, Gamelin). *Let Y be a complex vector space. Let A be an algebra of functions on Y such that the restriction of each $f \in A$ to any finite dimensional subspace of Y is an analytic polynomial. Let I be a proper ideal in A. Then there is a net (y_α) in Y such that $f(y_\alpha) \to 0$ for all $f \in I$.*

Доведення. Suppose that the conclusion fails. Then there are $(f_1, \dots, f_n) \in I$ such that
$$\max(|f_1(y)|, \dots, |f_n(y)|) \geq 1, \qquad y \in Y.$$
Let F be the map from Y to $\mathbb{C}^n$ having components $f_1, \dots, f_n$. Let V be an algebraic variety which does not contain 0. Hence there is a polynomial p on $\mathbb{C}^n$ such that $p = 0$ on V and $p(0) = 1$. Since the functions p together with the coordinate functions $z_1, \dots, z_n$ have no common zero, the ideal they generate in the polynomial ring on $\mathbb{C}^n$ is not proper (by the Hilbert Nullstellensatz). So there exist polynomials $q_0, q_1, \dots, q_n$ on $\mathbb{C}^n$ such that
$$pq_0 + z_1 q_1 + \dots + z_n q_n = 1 \qquad \text{on } \mathbb{C}^n,$$
implying
$$z_1 q_1 + \dots + z_n q_n = 1 \qquad \text{on } V.$$
Now let $g_1, \dots, g_n \in A$ be the compositions of $q_1, \dots, q_n$ respectively with F. Then $f_1 g_1 + \dots + f_n g_n = 1$, and the ideal I is not proper. □

7. *Let ϕ be any (possibly discontinuous) complex-valued homomorphism of $H_b(X)$. Then there is a net (x_α) in x such that $P(x_\alpha) \to \phi(P)$ for all analytic polynomials P on X.*

For a given uniform algebra A of continuous functions on a Banach space X we define an A-*topology* on X as the weakest topology such that all functions of A are continuous. That is A-topology is the restriction of the Gelfand topology to X. We say that a net x_α is A-convergent (notation $x_\alpha \xrightarrow{A} \phi$) if $f(x_\alpha)$ is convergent for every $f \in A$.

8. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. Then for every bounded $\mathcal{P}_0$-convergent net $(x_\alpha) \in X$ there is a continuous complex-valued homomorphism ϕ on $\mathcal{P}_0(X)$ such that $P(x_\alpha) \to \phi(P)$ for each $P \in \mathcal{P}_0(X)$.*

Доведення. It is easy to see that
$$\phi(P) := \lim_\alpha P(x_\alpha)$$
is a complex-valued homomorphism on $\mathcal{P}_0(X)$. From the boundedness of x_α it follows that ϕ is continuous. □

9. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ with unity which contains all finite type polynomials. Let J be an ideal in $\mathcal{P}_0(X)$ which is generated by a finite number of polynomials $P_1, \dots, P_n \in \mathcal{P}_0(X)$. If the polynomials $P_1, \dots, P_n$ have no common zeros, then J is not proper.*

Доведення. According to Lemma 6 there exists a finite dimensional subspace $Y_0 = \mathbb{C}^m \subset X$ such that $F(Y_0)^- = F(X)^-$, where $F(x) = (P_1(x), \dots P_n(x))$. Let $e_1, \dots, e_m$ be a basis in Y_0 and $e_1^*, \dots, e_m^*$ be the coordinate functionals. Denote by $P_k |_{Y_0}$ the restriction of P_k to Y_0. Since $\dim Y_0 = m < \infty$, there exists a continuous projection $T : X \to Y_0$. So any polynomial $Q \in \mathcal{P}(Y_0)$ can be extended to a polynomial $\widehat{Q} \in \mathcal{P}_0(X)$ by formula $\widehat{Q}(x) = Q(T(x))$. $\widehat{Q}$ belongs to $\mathcal{P}_0(X)$ because it is a finite type polynomial. Let us consider the map

$$G(x) = (P_1(x), \dots, P_n(x), \widehat{e_1^*}(x), \dots, \widehat{e_m^*}(x)) : X \to \mathbb{C}^{m+n}.$$

By definition of G, $G(X)^- = G(Y_0)^-$.

Suppose that J is a proper ideal in $\mathcal{P}_0(X)$ and so J is contained in a maximal ideal J_M. Let ϕ be a complex homomorphism such that $J_M = \ker \phi$. By Theorem 8 there exists a $\mathcal{P}_0$-convergent net (x_α) such that $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $P \in \mathcal{P}_0(X)$. Since $G(X)^- = G(Y_0)^-$, there is a net $(z_\beta) \subset Y_0$ such that $\lim_\alpha G(x_\alpha) = \lim_\beta G(z_\beta)$. Note that each polynomial $Q \in \mathcal{P}(Y_0)$ is generated by the coordinate functionals. Thus $\lim_\beta Q(z_\beta) = \lim_\alpha \widehat{Q}(x_\alpha) = \phi(Q)$. Also $\lim_\beta P_k |_{Y_0} (z_\beta) = \lim_\alpha P_k(x_\alpha) = \phi(P_k)$, $k = 1, \dots, n$. On the other hand, every $\mathcal{P}_0$-convergent net on a finite dimensional subspace is weakly convergent and so it converges to a point $x_0 \in Y_0 \subset X$. Thus $P_k(x_0) = 0$ for $1 \le k \le n$ that contradicts the assumption that $P_1, \dots, P_n$ have no common zeros. $\square$

Note that we also prowed that each complex homomorphism $\phi\colon \mathcal{P}_0(X) \to \mathbb{C}$ is a local evaluation. It means given $P_1, \dots, P_n \in \mathcal{P}_0(X)$, there exists $x_0 \in X$ such that $\phi(P_k) = P_k(x_0)$ for $k = 1, \dots, n$.

For an ideal $J \in \mathcal{P}_0(X)$, $V(J)$ denotes the *zero* of J, that is, the common set of zeros of all polynomials in J. Let G be a subset of X. Then $I(G)$ denotes the *hull* of G, that is, a set of all polynomials in $\mathcal{P}_0(X)$ which vanish on G. The set Rad$J$ is called the *radical* of J if $P^k \in J$ for some positive integer k implies $P \in$ Rad$J$. P is called a *radical* polynomial if it can be represented by a product of mutually different irreducible polynomials. In this case $(P) = \mathrm{Rad}(P)$.

A subalgebra A_0 of an algebra A is called *factorial* if for every $f \in A_0$ the equality $f = f_1 f_2$ implies that $f_1 \in A_0$ and $f_2 \in A_0$.

Using a standard idea from Algebraic geometry, now we can prove the next theorem which is a generalization of the well known Hilbert Nullstellensatz for algebras of polynomials of infinitely many variables.

10. *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and J be an ideal $\mathcal{P}_0(X)$ which is generated by a finite sequence of polynomials $P_1, \dots, P_n$. Then* $\mathrm{Rad}J \subset \mathcal{P}_0(X)$ *and*

$$I[V(J)] = \mathrm{Rad}J$$

in $\mathcal{P}_0(X)$.

Доведення. Since $\mathcal{P}_0(X)$ is factorial, $\mathrm{Rad}J \subset \mathcal{P}_0(X)$ for every ideal $J \subset \mathcal{P}_0(X)$. Evidently, $I[V(J)] \supset \mathrm{Rad}J$. Let $P \in \mathcal{P}_0(X)$ and $P(x) = 0$ for every $x \in V(J)$. Let $y \in \mathbb{C}$ be an additional "independent variable" which is associated with a basis vector e of an extra dimension. Consider a Banach space $X \oplus \mathbb{C}e = \{x + ye \colon x \in X, y \in \mathbb{C}\}$. We denote by $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ the algebra of polynomials on $X \oplus \mathbb{C}e$ such that every polynomial in $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ belongs to $\mathcal{P}_0(X)$ for arbitrary $y \in \mathbb{C}$. The polynomials $P_1, \dots, P_n, Py - 1$ have no common zeros. By Theorem 9 there are polynomials $Q_1, \dots, Q_{n+1} \in \mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ such that

$$\sum_{i=1}^{n} P_i Q_i + (Py - 1) Q_{n+1} \equiv 1.$$

Since it is an identity it will be still true for all vectors x such that $P(x) \neq 0$ if we substitute $y = 1/P(x)$. Thus

$$\sum_{i=1}^{n} P_i(x) Q_i(x, 1/P(x)) = 1.$$

Taking a common denominator, we find that for some positive integer N,

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) P^{-N}(x) = 1$$

or

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) = P^N(x), \tag{10}$$

where $Q'(x) = Q(x, P^{-1}) P^N(x) \in \mathcal{P}_0(X)$. The equality (10) holds on an open subset $X \setminus \ker P$, so it holds for every $x \in X$. But it means that P^N belongs to J. So $P \in \mathrm{Rad}J$. □

9. *Suppose* $\ker P$, $P \in \mathcal{P}(X)$ *contains a linear subspace Z of codimension one. Then there exists a polynomial $Q \in \mathcal{P}(X)$ and a linear functional L such that $P = QL$.*

Доведення. Let L be a linear functional on X such that $\ker L = Z$. By Theorem 10 L divides P^N for some positive integer N. So L divides P. $\square$

10. *Suppose* $\ker P$, $P \in \mathcal{P}(X)$ *is a union of a finite numbers of linear subspaces. Then P is a product of a finite numbers of linear functionals.*

Доведення. From the Hahn-Banach Theorem it follows that $\ker P$ is contained in a finite union of one codimensional linear subspaces. So P is factor of a product of linear functionals. Thus P is a product of a finite numbers of linear functionals. $\square$

11. *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and has the following property: If $Q \in \mathcal{P}_0(X)$ and $Q = Q_1 + \cdots + Q_n$ is the (necessary unique) representation of Q by homogeneous polynomials, then all Q_k there are in $\mathcal{P}_0(X)$. If P is continuous in the weakest topology on X such that all polynomials in $\mathcal{P}_0(X)$ are continuous, then $P \in \mathcal{P}_0(X)$.*

Доведення. Without loss of the generality, we can assume that P is m-homogeneous and irreducible. By the conditions of the theorem P must be bounded on a set $\{x \in X : |P_1(x)| < 1, \ldots, |P_n(x)| < 1\}$ for some $P_1, \ldots, P_n \in \mathcal{P}_0(X)$. Let J be an ideal generated by $P_1, \ldots, P_n$. If $x_0 \in V(J)$, then $tx_0 \in V(J)$ for every number t. So P is bonded on the subspace $tx_0, t \in \mathbb{C}$. But this is possible only if P is an identical zero on this subset. Hence $V(J) \subset \ker P$. Denote by A_0 a minimal factorial algebra which contains $\mathcal{P}_0(X)$ and P. By Theorem 10 there are $Q_1, \ldots, Q_n \in A_0$ such that

$$P_1Q_1 + \cdots + P_nQ_n = P.$$

We can assume that Q_k, $k = 1, \ldots, n$ are homogeneous and

$$\begin{cases} \deg Q_k + \deg P_k = m & \text{if} \quad \deg P_k \le m \\ Q_k = 0 & \text{if} \quad \deg P_k > m. \end{cases}$$

Indeed, let $Q_k = \sum_j Q_k^j$ is the decomposition of Q_k by j-homogeneous polynomials. Then

$$\sum_{k=1}^n P_kQ_k = \sum_{k=1}^n P_kQ_k^{m-\deg P_k} + \sum_{k=1}^n P_k \sum_{j \ne m-\deg P_k} Q_k^j = P$$

Since

$$\sum_{k=1}^{n} P_k \sum_{j \neq m-\deg P_k} Q_k^j$$

contains no m-homogeneous polynomials and $\deg P = m$,

$$\sum_{k=1}^{n} P_k \sum_{j \neq m-\deg P_k} Q_k^j = 0.$$

Putting $Q_k = Q_k^{m-\deg P_k}$, we have the required restrictions for Q_k. Since P is irreducible and $\deg Q_k < \deg P = m$, Q_k belongs to $\mathcal{P}_0(X) \subset A_0$ for every k. Therefore $P \in \mathcal{P}_0(X)$. $\square$

We say a set $\mathcal{V}$ is an *algebraic set of finite type* if $\mathcal{V}$ is the set of common zeros of some finite number of polynomials $P_1, \ldots, P_n \in \mathcal{P}(X)$. $\mathcal{V}$ is called an *algebraic variety of finite type* if the ideal $(P_1, \ldots, P_n)$ is prime.

Let $\mathcal{V} = V(P_1, \ldots, P_n)$ be an algebraic set of finite type. We can define an algebra of polynomials on $\mathcal{V}$ as a quotient algebra $\mathcal{P}(\mathcal{V}) := \mathcal{P}(X)/I(\mathcal{V})$. From Theorem 10 it follows that P is the identical zero in $\mathcal{P}(\mathcal{V})$ if and only if $P^N \in (P_1, \ldots, P_n)$ for some N and $\mathcal{P}(\mathcal{V})$ is an integral domain if and only if $(P_1, \ldots, P_n)$ is prime.

12. *Let ϕ be a complex homomorphism (possible discontinuous) of $\mathcal{P}(\mathcal{V})$. Then there is a net $(x_\alpha) \subset \mathcal{V}$ such that $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $P \in \mathcal{P}(\mathcal{V})$.*

Доведення. Note first that each complex homomorphism of $\mathcal{P}(\mathcal{V})$ is a local evaluation at $\mathcal{V}$. Indeed, if ϕ is a complex homomorphism of $\mathcal{P}(\mathcal{V})$, then ϕ may be considered as a complex homomorphism of $\mathcal{P}(X)$ which vanishes on $I(\mathcal{V})$. As we have indicated, ϕ must be a local evaluation at points of x, that is, for every polynomials $P_1, \ldots, P_n \in \mathcal{P}(X)$ there exists $x_0 \in X$ such that $\phi(P_k) = P_k(x_0)$. Since ϕ vanishes on $I(\mathcal{V})$, $x_0 \in \mathcal{V}$. Thus for every $Q_1, \ldots, Q_n \in \mathcal{P}(\mathcal{V})$ there exists $x_0 \in \mathcal{V}$ such that $\phi(Q_k) = Q_k(x_0)$, $1 \le k \le n$.

Consider the set of zeros of all finitely generated ideals in $\mathcal{P}(\mathcal{V})$:

$$\Big\{ V_\alpha = \bigcap_{k=1}^{m} \ker[P_{\alpha,k} - \phi(P_{\alpha,k})] \colon P_{\alpha,k} \in \mathcal{P}(X) \Big\}.$$

Each V_α is nonempty and The set $\{V_\alpha\}$ is naturally ordered by inclusion. Let $(x_\alpha) \subset \mathcal{V}$ be a net such that $x_\alpha \in V_\alpha$. It is clear, $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $P \in \mathcal{P}(\mathcal{V})$. □

1.4. **Applications for Symmetric Polynomials.** Let $\mathcal{G}$ be a group of linear isometries of X. A subset V of X is said to be $\mathcal{G}$-*symmetric* if it is invariant under the action of $\mathcal{G}$ on X. A function with a $\mathcal{G}$-symmetric domain is $\mathcal{G}$-*symmetric* if $f(\sigma(x)) = f(x)$ for every $\sigma \in \mathcal{G}$. It is clear that the kernel of a $\mathcal{G}$-symmetric polynomial is $\mathcal{G}$-symmetric. We consider the question: under which conditions a polynomial with a $\mathcal{G}$-symmetric set of zeros is $\mathcal{G}$-symmetric?

First we observe that if $P(x)$ is an irreducible polynomial then $P(\sigma(x))$ is irreducible for every $\sigma \in \mathcal{G}$. Indeed, if $P(\sigma(x)) = P_1(x)P_2(x)$, then

$$P(x) = P_1(\sigma^{-1}(x))P_2(\sigma^{-1}(x)).$$

Recall that a group homomorphism of $\mathcal{G}$ to $S^1 = \{e^{i\vartheta} : 0 \le \vartheta < 2\pi\}$ is called a *character* of $\mathcal{G}$.

11. *Suppose $\mathcal{G}$ has no nontrivial characters. If P is radical and* $\ker P$ *is a $\mathcal{G}$-symmetric set, then P is a $\mathcal{G}$-symmetric polynomial.*

Доведення. Since $\ker P = \ker P \circ \sigma$ for every $\sigma \in \mathcal{G}$, then, by Theorem 10, $P = cP \circ \sigma$ for some constant c. Because σ is an isometry, $|c| = 1$. If $c \ne 1$, then $c = c(\sigma)$ is a nontrivial character of $\mathcal{G}$. So $c = 1$. □

Suppose, for example $\mathcal{G} = S^1$, that is, the group of actions $x \rightsquigarrow e^{i\vartheta}x$. Then a homogeneous polynomial is $\mathcal{G}$-symmetric only if it is a constant. However, zero set of any homogeneous polynomial is S^1-symmetric.

Note that the subset of all $\mathcal{G}$-symmetric polynomials is a subalgebra in $\mathcal{P}(X)$.

13. *Suppose that the algebra of $\mathcal{G}$-symmetric polynomials on X is factorial and $\mathcal{G}$ has no nontrivial characters. Then the kernel of a $\mathcal{G}$-symmetric polynomial P is $\mathcal{G}$-symmetric if and only if P is $\mathcal{G}$-symmetric.*

Доведення. Let $P = P_1^{k_1} \dots P_n^{k_n}$, where $P_1, \dots, P_n$ are mutually different irreducible polynomials. Then $P_1 \dots P_n$ has the same set of zero that P. So if $\ker P$ is $\mathcal{G}$-symmetric, then by Proposition 11, $P_1 \dots P_n$ is $\mathcal{G}$-symmetric. By the assumption of the theorem, all polynomials $P_1, \dots, P_n$ must be $\mathcal{G}$-symmetric. So P is $\mathcal{G}$-symmetric as well. □

Note that if there exist a $\mathcal{G}$-symmetric polynomial $P = P_1P_2$ such that P_1 is not $\mathcal{G}$-symmetric, then $P_1^2P_2$ is a not $\mathcal{G}$-symmetric polynomial with a $\mathcal{G}$-symmetric kernel.

If X is the infinite-dimensional space ℓ_p, $1 \le p < \infty$ and $\mathcal{G}$ is the group of permutations of basis elements, then it is not difficult to see that the algebra of $\mathcal{G}$-symmetric polynomial is factorial and $\mathcal{G}$ has no nontrivial characters. For any finite-dimensional space there exists a nonsymmetric polynomial which has a symmetric kernel. For example $P(x) = x_1^2x_2 \dots x_n$ has a symmetric kernel in $\mathbb{C}^n$ but is not symmetric if $n > 1$.

Note that the algebra $\mathcal{P}_s(\ell_p)$ of symmetric polynomials on ℓ_p with respect to the group of permutations of basis elements $(e_k) \subset \ell_p$ does not satisfy the conditions of Theorem 10. However, this theorem is still true for this algebra. For simplicity we consider the case of ℓ_1 space.

14. *The elementary symmetric polynomials* $(R_i)_{i=1}^n$,

$$R_i(x) = \sum_{k_1 < \dots < k_i} x_{k_1} \dots x_{k_i},$$

where $x = \sum x_ie_i \in \ell_1$ *form an algebraic basis in* $\mathcal{P}_s(\ell_1)$. *It means that every symmetric polynomial* $Q \in \mathcal{P}_s(\ell_1)$ *can be represented by the way*

$$Q(x) = q(R_1(x), \dots, R_n(x)), \tag{11}$$

where q *is a polynomial in* $\mathcal{P}(\mathbb{C}^n)$ *and* $(R_k)_{k=1}^\infty$ *are algebraically independent, that is, if* $p(R_1(x), \dots, R_n(x)) \equiv 0$ *for some* $p \in \mathcal{P}(\mathbb{C}^n)$, *then* $p \equiv 0$.

Доведення. It is well known from Algebra (see [36]) that for any symmetric polynomial $Q^{(m)} \in \mathcal{P}_s(\mathbb{C}^m)$, $\deg Q^{(m)} = n$ there i a polynomial $q \in \mathcal{P}(\mathbb{C}^n)$ such that

$$Q^{(m)}(x) = q(R_1^{(m)}(x), \dots, R_n^{(m)}(x)),$$

where

$$R_i^{(m)}(x) = \sum_{k_1 < \dots < k_i}^{m} x_{k_1} \dots x_{k_i}.$$

Let $V_m = \mathrm{span}(e_1, \dots, e_n) \subset \ell_1$. We set

$$T_m \colon \sum_{i=1}^\infty x_ie_i \mapsto \sum_{i=1}^n x_ie_i$$

the projection from ℓ_1 to V_m. Let $Q \in \mathcal{P}_s(\ell_1)$, $\deg Q = n$. Then there exists a polynomial $q \in \mathcal{P}(\mathbb{C}^n)$ such that for every $m \geq n$ and for every $x \in \ell_1$

$$Q(T_m(x)) = q(R_1^{(m)}(x), \dots, R_n^{(m)}(x)).$$

Taking the limit as $m \to \infty$ we will get (11).

To show that R_j are algebraically independent, we observe that for every $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ there exists a vector $x_\xi = (x_1, \dots, x_n, 0, 0 \dots) \in \ell_1$ such that

$$R_1(x_\xi) = \xi_1, \dots, R_n(x_\xi) = \xi_n. \tag{12}$$

Indeed, according to the Vieta formula, the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \dots (-1)^n \xi_n = 0$$

satisfy the conditions $R_i(x_1, \dots, x_n) = \xi_i$ and so $x_\xi = (x_1, \dots, x_n)$ is as required.

If $p(\xi_1, \dots, \xi_n) \neq 0$ for some $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, then

$$P(R_1(x_\xi), \dots, R_n(x_\xi)) \neq 0.$$

□

12. *Let* $P_1, \dots, P_m \in \mathcal{P}_s(\ell_1)$ *be such that* $\ker P_1 \cap \dots \cap \ker P_m = \emptyset$. *Then there are* $Q_1, \dots, Q_m \in \mathcal{P}_s(\ell_1)$ *such that*

$$\sum_{i=1}^m P_i Q_i \equiv 1.$$

Доведення. Let $n = \max_i(\deg P_i)$. We may assume that

$$P_i(x) = g_i(R_1(x), \dots, R_n(x))$$

for some $g_i \in \mathcal{P}(\mathbb{C}^n)$. Let us suppose that at some point $\xi \in \mathbb{C}^n$, $\xi = (\xi_1, \dots, \xi_n)$, $g_i(\xi) = 0$. Then there is $x_\xi \in \ell_1$ such that $R_i(x_0) = \xi_i$ (see formula 12). So the common set of zeros of all g_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $q_1, \dots, q_m$ such that $\sum_i g_i q_i \equiv 1$. Put $Q_i(x) = q_i(R_1(x), \dots, R_n(x))$. □

2. Algebras of Analytic Functions

2.1. **Introduction to Analytic Functions.** Ω is *finitely open subset* of a Banach space X if for any finite dimensional affine subspace E of X, endowed with the Euclidean topology, $E \cap \Omega$ is open in E.

Definition 8. *We say that a map $f : \Omega \to Y$ is G-analytic (Gâteaux-analytic), and write $f \in H_G(\Omega, Y)$, if the restriction of f onto $E \cap \Omega$ is analytic for any finite-dimensional affine subspace E (or, equivalently, for any complex line $E \in X$). A G-analytic map defined on an open subset $\Omega \subset X$ to Y is called analytic, written $f \in H(\Omega, Y)$, if it is continuous.*

Every analytic function $f \in H(\Omega, Y)$ can be locally represented by its Taylor's series expansion

$$f(a+x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n f(a)(x, \dots, x)$$

which converges uniformly on a neighborhood of $a \in \Omega$, where $d^n f(a)(x, \dots, x) \in \mathcal{P}(^n X)$ is the nth Fréchet derivation of f at a by the direction $(x, \dots, x)$.

13. *Let f_k be a sequence of continuous k-homogeneous polynomials from X to Y. A necessary and sufficient condition for existence of $f \in H(X, Y)$ such that $f_k = d^k f(0)$ is that for any given $\epsilon > 0$ each $x \in X$ has a neighborhood U such that $\sup_U \|f_k\|^{1/k} \le \epsilon$ for k large enough.*

Let $f \in H(\Omega, Y)$, where Ω is an open subset of X, and $x \in \Omega$. The *radius of uniform convergence* $\varrho_x(f)$ of f at x is defined as supremum of λ, $\lambda \in \mathbb{C}$ such that $x + \lambda B \subset \Omega$ and the Taylor series of f at x converges to f uniformly on $x + \lambda B$, where B is the unit ball of X. The *radius of boundedness* of f at x is defined as supremum of λ, $\lambda \in \mathbb{C}$ such that f is bounded on $x + \lambda B$.

15. *The radius of uniform convergence of f at x coincides with the radius of boundedness of f at x and if $f \in H(X, Y)$, then*

$$\varrho_0(f) := \left(\limsup_{n\to\infty} \|f_n\|^{1/n} \right)^{-1},$$

where $f_n = d^k(x) f / n!$.

Denote by $H_b(X)$ the space of entire functions of *bounded type* that consists of entire functions on X which are bounded on bounded subsets (i.e. have the radius of boundedness equal to infinity). Note that if X is an infinite dimensional Banach space, then there exists a $\mathbb{C}$-valued entire function on X, f, such that $\varrho(f) < \infty$ for every $x \in X$ (see e.g. [19], p.169). The space $H_b(X)$ is a Fréchet algebra endowed with topology, generated by seminorms

$$\|f\|_r = \sup\{|f(x)| : x \in X, \|x\| < r\},$$

where $r > 0$ is a rational number.

Each linear functional $\phi \in H_b(X)'$ is continuous with respect to the norm of uniform convergence on some ball in X. The *radius function* $R(\phi)$ of ϕ is defined as infimum of all $r > 0$ such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB.

Denote by ϕ_n the restriction of ϕ to the subspace of n-homogeneous polynomials $\mathcal{P}(^nX)$. Then ϕ_n is a continuous linear functional on $\mathcal{P}(^nX)$ and

$$\|\phi_n\| = \sup\{\phi(P) : P \in \mathcal{P}(^nX), \|P\| \leq 1\}.$$

16. *The radius function R on $H_b(X)'$ is given by*

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Доведення. Let ϕ_n be the restriction of ϕ to $\mathcal{P}(^nX)$ and

$$\|\phi_n\| = \sup\{|\phi_n(P)| : P \in \mathcal{P}(^nX) \text{ with } \|P\| \leq 1\}.$$

Suppose that

$$0 < t < \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Then there is a sequence of homogeneous symmetric polynomials P_j of degree $n_j \to \infty$ such that $\|P_j\| = 1$ and $|\phi(P_j)| > t^{n_j}$. If $0 < r < t$, then by homogeneity,

$$\|P_j\|_r = \sup_{x\in rB} |P_j(x)| = r^{n_j},$$

so that

$$|\phi(P_j)| > (t/r)^{n_j} \|P_j\|_r,$$

and ϕ is not continuous on with respect to the norm of uniform convergence on rB. It follows that $R(\phi) \geq r$, and on account of the arbitrary choice of r we obtain

$$R(\phi) \geq \limsup_{n\to\infty} \|\phi_n\|^{1/n}$$

Let now $s > \limsup_{n\to\infty} \|\phi_n\|^{1/n}$ so that $s^m \geq \|\phi_m\|$ for m large. Then there is $c \geq 1$ such that $\|\phi_m\| \leq cs^m$ for every m. If $r > s$ is arbitrary and $f \in H_b(X)$ has Taylor series expansion $f = \sum_{n=1}^{\infty} f_n$, then

$$r^m \|f_m\| = \|f_m\|_r \leq \|f\|_r, \quad m \geq 0.$$

Hence
$$|\phi(f_m)| \leq \|\phi_m\|\|f_m\| \leq \frac{cs^m}{r^m}\|f\|_r$$
and so
$$\|\phi(f)\| \leq c\Big(\sum \frac{s^m}{r^m}\Big)\|f\|_r.$$
Thus ϕ is continuous with respect to the uniform norm on rB, and $R(\phi) \leq r$. Since r and s are arbitrary,
$$R(\phi) \leq \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

□

17. *Suppose that $\phi_n \in \mathcal{P}(^nX)'$ for $n \geq 0$, and suppose that norm of ϕ_n satisfy*
$$\|\phi_n\| \leq cs^n$$
for some $c, s > 0$. Then there is a unique $\phi \in H_b(X)'$ whose restriction to $\mathcal{P}(^nX)$ coincides with ϕ_n, $n \geq 0$.

The next theorem easily follows from Theorem 5.

18. *Let $f \in H_b(X)$ and $f = \sum f_n$ is its Taylor series. Then there exists $\widetilde{f} \in H_b(X'')$ with the Taylor series expansion $\widetilde{f} = \sum \widetilde{f}_n$ such that $\widetilde{f}_n$ is the Aron-Berner extension of f_n. Moreover, $\|\widetilde{f}\| = \|f\|$ and the operator $f \mapsto \widetilde{f}$ is a homomorphism between the Fréchet algebras $H_b(X)$ and $H_b(X'')$.*

2.2. **The Spectrum of H_b.** Let us denote by $A_n(X)$ the closure of the algebra, generated by polynomials from $\mathcal{P}(^{\leq n}X)$ with respect to the uniform topology on bounded subsets. It is clear $A_1(X) \cap \mathcal{P}(^nX) = \mathcal{P}_A(^nX)$ and $A_n(X)$ is a Fréchet algebra of entire analytic functions on X for every n. The closure of the algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on bounded subsets coincides with $H_b(X)$. The closure of the algebra of all polynomials with respect to the uniform topology on the unit ball B, $H^\infty_{uc}(B)$ is the algebra of all analytic functions on B which are uniformly continuous on B. We will use short notations M_b and M_{uc} for the spectra $M(H_b(X))$ and $M(H^\infty_{uc}(B))$ respectively.

14. *Let $\phi \in H_b(X)'$ such that $\phi(P) = 0$ for every $P \in \mathcal{P}(^mX) \cap A_{m-1}(X)$, where m is a fixed positive integer and $\phi_m \neq 0$. Then there is $\psi \in M_b$ such that $\psi_k = 0$ for $k < m$ and $\psi_m = \phi_m$. The radius function $R(\psi) = \|\phi_m\|^{1/m}$.*

Доведення. Since $\phi_m \neq 0$, there is an element $w \in (\otimes^m_{s,\pi} X)''$, $w \neq 0$ such that for any m-homogeneous polynomial P, $\phi(P) = \phi_m(P) = \widetilde{P}_{(m)}(w)$, where $\widetilde{P}_{(m)}$ is the Aron-Berner extension of linear functional $P_{(m)}$ from $\otimes^m_{s,\pi} X$ to $(\otimes^m_{s,\pi} X)''$ and $\|w\| = \|\phi_m\|$. For an arbitrary n-homogeneous polynomial Q we set

$$(13) \qquad \psi(Q) = \begin{cases} \widetilde{Q}_{(m)}(w) & \text{if } n = mk \text{ for some } k \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\widetilde{Q}_{(m)}$ is the Aron-Berner extension of the k-homogeneous polynomial $Q_{(m)}$ from $\otimes^m_{s,\pi} X$ to $(\otimes^m_{s,\pi} X)''$.

Let (u_α) be a net from $\otimes^m_{s,\pi} X$ that converges to w in the weak-star topology of $(\otimes^m_{s,\pi} X)''$, where α belongs to an index set $\mathfrak{A}$. We can assume that u_α has a representation $u_\alpha = \sum\limits_{j=1}^{\infty} x_{j,\alpha}^{\otimes m}$ for some $x_{j,\alpha} \in X$. Let us show that $\psi(PQ) = \psi(P)\psi(Q)$ for any homogeneous polynomials P and Q. Let us suppose first that $\deg(PQ) = mr + l$ for some integers $r \geq 0$ and $m > l > 0$. Then P or Q has degree equal to $mk + s$, $k \geq 0$, $m > s > 0$. Thus, by the definition, $\psi(PQ) = 0$ and $\psi(P)\psi(Q) = 0$. Suppose that $\deg(PQ) = mr$ for some integer $r \geq 0$. If $\deg P = mk$ and $\deg Q = mn$ for $k, n \geq 0$, then $\deg(PQ) = m(k+n)$ and $\psi(PQ) = (\widetilde{PQ})_{(m)}(w) = \widetilde{P}_{(m)}(w)\widetilde{Q}_{(m)}(w) = \psi(P)\psi(Q)$.

Let now $\deg P = mk + l$ and $\deg Q = mn + r$, $l, r > 0$, $l + r = m$. Write $\nu = 1/(\deg P + \deg Q)! = 1/(m(k+n+1))!$. Let A_{PQ} denote the symmetric multilinear map, associated with PQ. Then

$$\begin{aligned} & A_{PQ}(x_1, \ldots, x_{m(k+n+1)}) \\ & = \nu \sum_{\sigma \in S_{m(k+n+1)}} A_P(x_{\sigma(1)}, \ldots, x_{\sigma(mk+l)}) A_Q(x_{\sigma(mk+l+1)}, \ldots, x_{\sigma(m(k+n+1))}), \end{aligned}$$

where $S_{m(k+n+1)}$ is the group of permutations on $\{1, \ldots, m(k+n+1)\}$. Thus for $\alpha_1, \ldots, \alpha_{k+n+1} \in \mathfrak{A}$ we have

$$\begin{aligned}
&\psi(PQ)\\
&\quad = (\widetilde{PQ})_{(m)}(w) = \lim_{\alpha_1,\dots,\alpha_{k+n+1}} \widetilde{A}_{PQ_{(m)}}(u_{\alpha_1},\dots,u_{\alpha_{k+n+1}})\\
&\quad = \lim_{\alpha_1,\dots,\alpha_{k+n+1}} \widetilde{A}_{PQ_{(m)}}\left(\sum_{j=1}^{\infty} x_{j,\alpha_1}^{\otimes m},\dots,\sum_{j=1}^{\infty} x_{j,\alpha_{k+n+1}}^{\otimes m}\right)
\end{aligned}$$

$$= \nu \sum_{\sigma\in S_{m(k+n+1)}} \lim_{\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k+n+1)}} \sum_{j_1,\dots,j_{k+n+1}=1}^{\infty} A_P(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},$$

$$x^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}})A_Q(x^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}},x^m_{j_{\sigma(k+2)},\alpha_{\sigma(k+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}).$$

Fix some $\sigma \in S_{m(k+n+1)}$ and fix all $x_{j_{\sigma(i)},\alpha_{\sigma(i)}}$, for $i \le k$ and for $i > k+1$. Then

$$\begin{aligned}
&\sum_{j_1,\dots,j_{k+n+1}=1}^{\infty} \lim_{\alpha_{\sigma(k+1)}} A_P(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},x^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}})\\
&\quad\times A_Q(x^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}},x^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}) = 0
\end{aligned}$$

because for fixed $x_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i \le k$

$$P_\sigma(y) := \sum_{j_1,\dots,j_k,j_{k+2},\dots,j_{k+n+1}=1}^{\infty} A_P(x^m_{j_{\sigma(1)},\alpha_{\sigma(1)}},\dots,x^m_{j_{\sigma(k)},\alpha_{\sigma(k)}},y^l)$$

is an l-homogeneous polynomial and for fixed $x_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i > k+1$

$$Q_\sigma(y) := \sum_{j_1,\dots,j_k,j_{k+2},\dots,j_{k+n+1}=1}^{\infty} A_Q(y^r,x^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}},\dots,x^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}})$$

is an r-homogeneous polynomial. Thus $P_\sigma Q_\sigma \in A_{m-1}(X)$. Hence

$$\lim_{\alpha}(P_\sigma Q_\sigma)_{(m)}(u_\alpha) = \psi(P_\sigma Q_\sigma) = 0$$

for every fixed σ. Thus $\psi(PQ) = 0$. On the other hand, $\psi(P)\psi(Q) = 0$ by the definition of ψ. So $\psi(PQ) = \psi(P)\psi(Q)$.

Thus we have defined the multiplicative function ψ on homogeneous polynomials. We can extend it by linearity and distributivity to a linear multiplicative functional on the algebra of all continuous polynomials $\mathcal{P}(X)$. If ψ_n is the restriction of ψ to $\mathcal{P}({}^nX)$, then $\|\psi_n\| = \|w\|^{n/m}$ if n/m is a positive integer and $\|\psi_n\| = 0$ otherwise. Hence $\psi = \sum_{n=0}^{\infty} \psi_n$ is a continuous linear multiplicative functional on $H_b(X)$ by Theorem 17 and the radius function of ψ can be computed by

$$R(\psi) = \limsup_{n\to\infty} \|\psi_n\|^{1/n} = \limsup_{n\to\infty} \|w\|^{n/mn} = \|w\|^{1/m} = \|\phi_m\|^{1/m}$$

as required. □

For each fixed $x \in X$, the translation operator T_x is defined on $H_b(X)$ by

$$(T_x f)(y) = f(y + x), \qquad f \in H_b(X).$$

It is not complicated to check that $T_x f \in H_b(X)$ and for fixed $\phi \in H_b(X)'$ the function $x \mapsto \phi(T_x f)$, $x \in X$, belongs to $H_b(X)$ (see [4]).

For fixed $\phi, \theta \in H_b(X)'$ the convolution product $\phi * \theta$ in $H_b(X)$ is defined by

$$(\phi * \theta)(f) = \phi(\theta(T_x f)), \qquad f \in H_b(X).$$

Let $\phi, \theta \in M_b$. By Corollary 7, there exist nets $(x_\alpha), (y_\beta) \subset X$ such that

$$\phi(P) = \lim_{\alpha} P(x_\alpha), \qquad \theta(P) = \lim_{\beta} P(y_\beta) \tag{14}$$

for every polynomial P. According to our notations, we will write the condition (14) by $x_\alpha \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \theta$. Thus for every polynomial P we have: $(\phi * \theta)(P) = \lim_\beta \lim_\alpha P(x_\alpha + y_\beta)$. Note that M_b is a semigroup with respect to the convolution product and $\phi * \theta \neq \theta * \phi$ in general (see [7, Remark 3.5]). We denote $\phi_1 * \cdots * \phi_n$ briefly by $\mathop{\ast}\limits_{k=1}^{n} \phi_k$.

Let I_k be the minimal closed ideal in $H_b(X)$, generated by all m-homogeneous polynomials, $0 < m \leq k$. Evidently, I_k is a proper ideal (contains no unit) so it is contained in a closed maximal ideal (see [31, p. 228]). Let

$$\Phi_k := \{\phi \in M_b \ : \ \ker \phi \supset I_k\}.$$

We set $\Phi_0 := M_b$. The functional $\delta(0)$, that is point evaluation at zero, belongs to Φ_k for every $k > 0$.

15. *If $A_m(X) \neq A_{m-1}(X)$ for some $m > 1$, then there exists $\psi \in \Phi_{m-1}$ such that $\psi \notin \Phi_m$.*

Доведення. Let $P \in \mathcal{P}(^m X)$ and $P \notin A_{m-1}(X)$. Since $A_{m-1}(X)$ is a closed subspace of $H_b(X)$, by the Hahn-Banach Theorem there exists a linear functional $\phi \in H_b(X)'$ such that $\phi(Q) = 0$ for every $Q \in A_{m-1}(X)$ and $\phi(P) \neq 0$. So $\phi_k \equiv 0$ for $k < m$ and $\phi_m(P) \neq 0$. By Lemma 14 there exists $\psi \in M_b$ such that $\psi_k = \phi_k$ for $k = 1, \ldots, m$. Thus $\psi \in \Phi_{m-1}$, but $\psi \notin \Phi_m$. □

Note that $A_1(c_0) = A_n(c_0)$ for every n, but $A_k(\ell_p) = A_m(\ell_p)$ for $k \neq m$ if and only if $k < p$ and $m < p$. Moreover, if X admits a polynomial which is not weakly sequentially continuous, then the chain of algebras $\{A_k(X)\}$ does not stabilize and if X contains ℓ_1, then $A_k(X) \neq A_m(X)$ for $k \neq m$ [26, **?**].

16. *If $\phi, \psi \in M_b$ and $\psi \in \Phi_{k-1}$, then $\phi * \psi(P) = \phi(P) + \psi(P)$ for every $P \in \mathcal{P}(^k X)$.*

Доведення. Let (x_α) and (y_β) be nets in X such that $x_\alpha \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \psi$. For any fixed y_β and $0 < n < k$, $A_P(x^{k-n}, y_\beta^n)$ is a $(k-n)$-homogeneous polynomial. Thus

$$\phi(A_P(x^{k-n}, y_\beta^n)) = \lim_\alpha A_P(x_\alpha^{k-n}, y_\beta^n) = 0.$$

Therefore,

$$\begin{aligned} \phi * \psi(P) &= \lim_{\beta,\alpha} P(x_\alpha + y_\beta) \\ &= \sum_{n+m=k} \lim_{\beta,\alpha} A_P(x_\alpha^n, y_\beta^m) = \sum_{n+m=k} \lim_\beta \Big(\lim_\alpha A_P(x_\alpha^n, y_\beta^m) \Big) \\ &= \lim_\beta \Big(\lim_\alpha A_P(x_\alpha, \ldots, x_\alpha) + A_P(y_\beta, \ldots, y_\beta) \Big) = \phi(P) + \psi(P). \end{aligned}$$

□

17. *If $P \in \mathcal{P}(^k X)$, $\phi_j \in \Phi_{j-1}$, then for every $m > k$, $\mathop{\ast}\limits_{j=1}^{m} \phi_j(P) = \mathop{\ast}\limits_{j=1}^{k} \phi_j(P)$.*

Доведення. Since $\phi_j \in \Phi_{j-1}$, $\phi_j(P) = 0$ for every $j > k$. □

Given a sequence $(\phi_n)_{n=1}^\infty \subset M_b$, $\phi_n \in \Phi_{n-1}$, the infinite convolution $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n$ denotes a linear multiplicative functional on the algebra of all polynomials $\mathcal{P}(X)$ such that $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n(P) = \mathop{\ast}\limits_{n=1}^{k} \phi_n(P)$ if $P \in \mathcal{P}(^k X)$

for an arbitrary k. This multiplicative functional uniquely determines a functional in M_b (which we denote by the same symbol $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n$) if it is continuous.

The point evaluation operator δ maps X into M_b by $x \mapsto \delta(x)$, $\delta(x)(f) = f(x)$. The operator $\widetilde{\delta}$ is the extension of δ onto X'', i.e. $\widetilde{\delta}(x'')(f) = \widetilde{f}(x'')$ for every $x'' \in X''$.

19. *There exists a sequence of dual Banach spaces $(E_n)_{n=1}^{\infty}$ and a sequence of maps $\delta^{(n)} : E_n \to M_b$ such that $E_1 = X''$, $E_n = \mathcal{P}(^nX)' \cap I_{n-1}^{\perp}$, $\delta^{(1)} = \tilde{\delta}$ and such that an arbitrary complex homomorphism $\phi \in M_b$ has a representation*

$$\phi = \mathop{\ast}\limits_{n=1}^{\infty} \delta^{(n)}(u_n) \tag{15}$$

for some $u_n \in E_n$, $n = 1, 2, \ldots$.

Доведення. Put $E_1 = X''$. Then $\delta^{(1)}(x'') = \tilde{\delta}(x'') \in M_b$ for every $x'' \in X''$. Suppose that spaces E_k and maps $\delta^{(k)}$ are constructed for $k < n$. Denote by E_n the set $\{\pi_n(\phi) : \phi \in \Phi_{n-1}\}$, where $\pi_n(\phi) = \phi_n$ is the restriction of ϕ onto subspace $\mathcal{P}(^nX)$. In other words, E_n consists of linear continuous functionals on $\mathcal{P}(^nX)$ that vanish on all polynomials in $\mathcal{P}(^nX) \cap A_{n-1}$. If $A_n = A_{n-1}$, then $E_n \equiv 0$. Otherwise, by Corollary 15, there are nonzero points in E_n.

By Lemma 16, for $P \in \mathcal{P}(^nX)$ and $\phi, \psi \in \Phi_{n-1} \subset M_b$, $\pi_n(\phi * \psi)(P) = \phi * \psi(P) = \phi(P) + \psi(P) = \pi_n\phi(P) + \pi_n\psi(P)$. Hence $\pi_n(\phi * \psi) = \pi_n(\phi) + \pi_n(\psi)$. For an arbitrary complex number a, $a\phi \in H_b(X)'$ and $\pi_k(a\phi) = a\pi_k(\phi)$. So $a\phi$ vanishes on all homogeneous polynomials of degree less than n. By Lemma 14 there exists $\psi \in M_b$ such that $\psi_k = a\phi_k$ for $1 \le k \le n$. Thus $\psi \in \Phi_{n-1}$ and $a\phi_n = \psi_n \in E_n$. Hence E_n is a linear space and polynomials from $\mathcal{P}(^nX)$ are acting on E_n as linear functionals. Put $W_n = \mathcal{P}(^nX)/(I_{n-1} \cap \mathcal{P}(^nX))$. Then W_n is a Banach space of linear functionals on E_n and the functionals from W_n separate points of E_n. Let us define a norm on E_n, $\|\cdot\|_n$ as the supremum of values of a vector from E_n on the unit ball of W_n. Therefore $W_n' = (\mathcal{P}(^nX)/(I_{n-1} \cap \mathcal{P}(^nX)))' = \mathcal{P}(^nX)' \cap I_{n-1}^{\perp} \supset E_n$. On the other hand, if $u \in \mathcal{P}(^nX)' \cap I_{n-1}^{\perp}$, then by Lemma 14 $u = \pi_n(\phi)$ for some $\phi \in M_b$ and so $u \in E_n$. Thus $E_n = W_n'$.

For given $w \in E_n$ let us define $\delta^{(n)}(w)(Q) = \psi(Q)$ on homogeneous polynomials Q by formula (13) and extend it to the unique complex homomorphism on $H_b(X)$ as in Lemma 14. So $\delta^{(n)}$ maps E_n into M_b.

For any $\phi \in M_b$ put $u_1 := \phi_1 \in X'' = E_1$, $u_2 := \phi_2 - \pi_2(\delta^{(1)}(u_1))$. It is clear that $u_2 \in E_2$. Suppose that we have defined $u_k \in E_k$, $k < n$. Set

$$u_n := \phi_n - \pi_n \left(\mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k) \right). \tag{16}$$

Let us show that $u_n \in E_n$. It is enough to check that for every $P \in \mathcal{P}({}^n X)$ such that $P = P_k P_m$, $\deg P_k = k \neq 0$, $\deg P_n = n \neq 0$ implies $u_n(P) = 0$. Note that for every n-homogeneous polynomials P_n,

$$\phi_n - \pi_n \left(\mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k) \right)(P_n) = \phi_n - \mathop{\ast}_{k=1}^{n-1} \delta^{(k)}(u_k)(P_n).$$

From the multiplicativity of ϕ and Lemma 17 it follows that

$$\begin{aligned}
u_n(P) &= \phi_n(P_k P_m) - \mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_k P_m) = \phi_k(P_k)\phi_m(P_m) \\
&- \left(\mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_k) \right) \left(\mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_m) \right) \\
&= \left(u_k(P_k) + \mathop{\ast}_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k) \right) \left(u_m(P_m) + \mathop{\ast}_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m) \right) \\
&- \left(\mathop{\ast}_{j=1}^{k} \delta^{(j)}(u_j)(P_k) \right) \left(\mathop{\ast}_{j=1}^{m} \delta^{(j)}(u_j)(P_m) \right) = 0.
\end{aligned}$$

The last equality holds because by the induction assumption, $u_k \in E_k$, $u_m \in E_m$ and hence, by Lemma 16,

$$u_k(P_k) + \mathop{\ast}_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k) = \mathop{\ast}_{j=1}^{k} \delta^{(j)}(u_j)(P_k) \tag{17}$$

and

$$u_m(P_m) + \mathop{\ast}_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m) = \mathop{\ast}_{j=1}^{m} \delta^{(j)}(u_j)(P_m).$$

Let us consider the functional $\mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$. Since $u_k \in E_k$, by Lemma 16,

$$\mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)(f) = f(0) + \sum_{n=1}^{\infty} \mathop{\ast}_{j=1}^{n} \delta^{(j)}(u_j)(f_n),$$

where $f = \sum f_n$ is the Taylor series expansion of f. Hence $\mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$ is well defined on $\mathcal{P}(X)$. On the other hand, applying (16) and (17) we obtain

$$\left(\phi - \mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)\right)(P_n) = \phi_n(P_n) - \mathop{\ast}_{j=1}^{n} \delta^{(j)}(u_j)(P_n)$$
$$= u_n(P) + \mathop{\ast}_{j=1}^{n-1} \delta^{(j)}(u_j)(P_n) - \mathop{\ast}_{j=1}^{n} \delta^{(j)}(u_j)(P_n) = 0$$

for arbitrary $P_n \in \mathcal{P}(^n X)$. Thus $\phi = \mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $\mathcal{P}(X)$. Hence $\phi = \mathop{\ast}_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $H_b(X)$. □

20. *Let $(u_k)_{k=1}^{\infty}$ be a sequence such that $u_k \in E_k$ for every k. Then $\phi = \mathop{\ast}_{k=1}^{\infty} \delta^{(k)}(u_k)$ is a continuous complex homomorphism in M_b if and only if $\sup_k \|u_k\|^{1/k} < \infty$. In this case*

$$\sup_k \|u_k\|^{1/k} \le R(\phi) \le e \sup_k \|u_k\|^{1/k}. \tag{18}$$

3. Applications

3.1. **Discontinuous Complex Homomorphisms. The Michael Problem.** E. Michael [30] posed the following problem in 1952 which is still open:

Is every complex homomorphism of a commutative Fréchet algebra continuous?

In [31, p. 240] Mujica proved that the The Michael Problem can be reduced to the case of the algebra $H_b(X)$ for an arbitrary Banach space X with a Schauder basis. However a dens subalgebra of $H_b(X)$ may admit a discontinuous complex homomorphism. Dixon [21] has given an example of an algebra of polynomials of infinitely many variables which admits discontinuous scalar-valued homomorphisms. In [23] Galindo et al. gave a construction of a discontinuous scalar-valued homomorphism of algebra of polynomials on arbitrary infinite-dimensional Banach space. Their idea is to take a discontinuous functional on X' and extend it to a functional on $\mathcal{P}(X)$. The next proposition shows that the restriction of a discontinuous complex homomorphism on $A_n(X) \cap \mathcal{P}(X)$ can be continuous for every n.

18. *If the sequence of algebras $A_n(X)$ does not stabilize, then there is a discontinuous complex homomorphism ζ on $\mathcal{P}(X)$ such that the restriction of ζ on $A_n(X) \cap \mathcal{P}(X)$ is a continuous complex homomorphism for every n.*

Доведення. By Corollary 15 and Theorem 19 there exists an infinity sequence $(u_k)_{k=1}^{\infty}$, $u_k \in E_k$, $u_k \neq 0$. Since each E_k is a linear space, we can choose u_k such that $\sup_k \|u_k\|_k^{1/k} = \infty$. Put $\zeta = \mathop{\ast}\limits_{k=1}^{\infty} \delta^{(k)}(u_k)$. Evidently, $\zeta(f) = \mathop{\ast}\limits_{k=1}^{n} \delta^{(k)}(u_k)(f)$ for every $f \in A_n(X)$. So ζ is well defined and continuous on $A_n(X) \cap \mathcal{P}(X)$. If ζ is continuous on $\mathcal{P}(X)$, then it can be extended to a continuous complex homomorphism on $H_b(X)$. But it contradicts Theorem 20. □

A discontinuous complex homomorphism of $H_b(X)$ (if it exists) eventually, need not to be discontinuous on $\mathcal{P}(X)$.

19. *If there exists a discontinuous complex homomorphism ϕ of $H_b(X)$, then there exists a discontinuous complex homomorphism ψ of $H_b(X)$ such that the restriction of ϕ on X' is discontinuous.*

Доведення. Let (f_n) be a sequence in $H_b(X)$ such that $\|f_n\|_r \to 0$ as $n \to \infty$ for every $r > 0$ and $\phi(f_n) > 4^n$. Let (e_n) be a normalized basis sequence in X with a normalized biorthogonal sequence $(e_n^*) \subset X$. Put

$$F(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x) e_n.$$

It is easy to check that $F \in H_b(X, X)$. So the composition operator $T_F \colon f \mapsto f \circ F$ is a continuous homomorphism from $H_b(X)$ to itself. We set $\psi := \phi \circ F$. Then ψ is a complex homomorphism of $H_b(X)$ and

$$|\psi(e_n^*)| = \left| \frac{\phi(f_n)}{2^n} \right| > 2^n.$$

□

3.2. **Homomorphisms.** Recall that $\mathbb{E}^n \subset \mathbb{E}^\infty \subset M_b$,

$$\mathbb{E}^n := E_1 \times \cdots \times E_n = \{(u_1, \ldots, u_n) \colon u_k \in E_k, 1 \leq k \leq n\}.$$

20. *Let Θ be a continuous homomorphism from $H_b(X)$ to itself. Then for every positive integer n there exists a map $F_n \colon \mathbb{E}^n \to \mathbb{E}^n$ such that for every $f \in A_n(X)$, $\Theta(f) = \widehat{f} \circ F_n$.*

Доведення. If $\mathfrak{u} = (u_1, \dots, u_n) \in \mathbb{E}^n$. Then $\phi_{\mathfrak{u}} \circ \Theta = \mathop{\ast}\limits_{k=1}^{m} \delta^{(k)}(u_k) \circ \Theta \in M_b$. By Theorem 19 there exists a point $\mathfrak{v} = (v_1, v_2, \dots) \in M_b$ such that $\phi_{\mathfrak{u}} \circ \Theta(f) = \widehat{f}(\mathfrak{v})$. If $f \in A_n(X)$, $\widehat{f}(\mathfrak{v}) = \widehat{f}((v_1, \dots, v_n))$. So we can assume that $\mathfrak{v} \in \mathbb{E}^n$. Put $F_n(\mathfrak{u}) := \mathfrak{v}$. Thus we have constructed the required mapping $\mathfrak{u} \mapsto F_n(\mathfrak{u})$ with the property $\Theta(f) = \widehat{f} \circ F_n$. $\square$

We notice that F_n need not to be analytic in $\mathbb{E}^n$. For example, let $0 \neq u_2 \in E_2$ and g be a linear functional on X. We define $F\colon X \to E_2$ by $F(x) := \sqrt{g(x)}u$. Then

$$\Theta_F(f)(x) := f \circ F(x) = \sum_{n=0}^{\infty} (g(x))^n f_{2n}(u_2),$$

for an arbitrary $f = \sum f_n \in H_b(X)$. It is easy to see that Θ_F is a continuous homomorphism of $H_b(X)$ to itself but F is not holomorphic.

A homomorphism Θ from $H_b(X)$ to itself is called *AB-composition homomorphism* [15] if there exists $F \in H_b(X'', X'')$ such that $\widetilde{\Theta(f)}(x'') = \widetilde{f}(F(x''))$, where $\widetilde{f}$ is the Aron-Berner extension of f.

21. *Every polynomial on X is approximable if and only if every homomorphism on $H_b(X)$ is an AB-composition homomorphism.*

Доведення. Suppose that every polynomial on X is approximable. Then $H_b(X) = A_1(X)$. By Proposition 20 for every homomorphism $\Theta\colon H_b(X) \to H_b(X)$ there exists a mapping $F\colon X'' \to X''$ such that $\Theta(f) = \widehat{f} \circ F = \widetilde{f} \circ F$. In particular, for every $f \in X'$, $\widetilde{f} \circ F \in H_b(X)$. So we can say that F is weak-star analytic map on X''. By a classical result of Dunford [22] and Grothendieck [28] on weak-star analytic mappings, F is analytic on X''. Since $\widetilde{f} \circ F$ is bounded on bounded sets of X'' for every $f \in X'$ and weak-star boundedness implies boundedness, $F \in H_b(X'', X'')$.

Suppose now that $A_n(X) \neq A_1(X)$ for some n. Let us choose $u_n \in E_n$ $u_n \neq 0$ and $l \in X'$, $l \neq 0$. Put $F(x) := l(x)u_n$ and $\Theta(f)(x) := \widehat{f}(F(x))$. Since $F \in H_b(X, \mathbb{E}^n)$, $\Theta(f)(x) \in H_b(X)$. But Θ is not an AB-composition homomorphism because $\Theta \not\equiv 0$ and $\Theta(f) = 0$ for every $f \in A_1$. $\square$

Since the approximation property of X' implies that every weakly continuous on bounded sets polynomial is approximable [8], we have the following corollary.

21. (c.f. [15]). *Let X' have the approximation property. Then every polynomial on X is weakly continuous on bounded sets if and only if every homomorphism on $H_b(X)$ is an AB-composition homomorphism.*

The result of Theorem 21 can be improved for a reflexive Banach space.

22. (Mujica [32]). *If $\mathcal{P}(X) = \mathcal{P}_A(X)$ for a reflexive Banach space X, then for every continuous homomorphism $\Theta\colon H_b(X) \to H_b(X)$ there is a unique map $F \in H_b(X, X)$ such that $\Theta(f) = f \circ F$.*

22. *Let X be a reflexive Banach space with $\mathcal{P}(X) = \mathcal{P}_A(X)$ and $F \in H_b(X, X)$. Suppose that $\Theta(f) = f \circ F$ is an isomorphism of $H_b(X)$. Then F is invertible and $F^{-1} \in H_b(X, X)$.*

Доведення. By Theorem 22 there exists a map $G \in H_b(X, X)$ such that $\Theta^{-1}(f) = f \circ G$. It is easy to see that $G = F^{-1}$. □

3.3. **Derivations.** Let $u_k \in E_k$. According to Theorem 19 we can define a complex homomorphism $\phi \in M_b = \delta^{(k)}(u_k)$ and $\phi(f) = \widehat{f}(u_k)$ for every $f \in H_b(X)$. However, u_k belongs to $(\otimes^k_{s,\pi} X)''$ and so there is an another natural way to define a linear functional on $H_b(X)$, associated with u_k. Let $\theta = \theta(u_k) = \sum \theta_m \in H_b(X)'$ such that $\theta_k(P) = \widehat{P}(u_k)$ if $P \in \mathcal{P}(^k X)$ and $\theta_m = 0$ if $m \neq k$. Recall that here θ_m is the restriction of θ to $\mathcal{P}(^k X)$. It is easy to see that θ is not a homomorphism if $u_k \neq 0$. We define a linear operator on $H_b(X)$, $\partial_{(k)}(u_k)$ by

$$\partial_{(k)}(u_k)(f)(x) := \theta(u_k) \circ T_x(f).$$

For the multilinear form A_P associated with an n-homogeneous polynomial P we denote by $\widehat{A_P}(x^{n-k}, u_k)$ the value of the Gelfand transform at $u_k \in E_k$ of the k-homogeneous polynomial $A_P(x^{n-k}, \cdot)$, where x is fixed.

23. *Let $u_k \in E_k$. Then the operator $\partial_{(k)}(u_k)$ is a continuous derivation on $H_b(X)$,*

$$\partial_{(k)}(u_k)(P)(x) = \binom{n}{k} \widehat{A_P}(x^{n-k}, u_k) \tag{19}$$

for every $P \in \mathcal{P}(^n X)$ and

$$\delta^{(k)}(u_k)(f)(x) = \sum_{m=0}^{\infty} \frac{(k!)^m}{(mk)!} \partial^m_{(k)}(u_k)(f)(x) \tag{20}$$

for every $f \in H_b(X)$.

Доведення. To prove formula (19) we observe that

$$P(z+x) = \sum_{m=0}^{n} \binom{n}{m} A_P(x^{n-m}, z^m).$$

So for a fixed x,

$$\partial_{(k)}(u_k)(P)(x) = \theta(u_k)(P(z+x)) = \binom{n}{k} \widehat{A_P}(x^{n-k}, u_k).$$

Note that if $\deg P \leq k$, then $\partial_{(k)}(u_k)(P)(x) = 0$ for every x by the definition of $\partial_{(k)}(u_k)$.

Let $P \in \mathcal{P}(^n X)$ and $Q \in \mathcal{P}(^m X)$. The multilinear form $A_{PQ}(x^{nm-k}, z^k)$ associated with PQ can be represented by

$$A_{PQ}(x^{nm-k}, z^k) = A^1_{PQ}(x^{nm-k}, z^k) + A^2_{PQ}(x^{nm-k}, z^k) + A^3_{PQ}(x^{nm-k}, z^k),$$

where

$$A^1_{PQ}(x^{n-k}, z^k) = A_P(x^{n-k}, z^k) A^Q(x^m);$$
$$A^2_{PQ}(x^{n-k}, z^k) = A_P(x^n) A^Q(z^k, x^{m-k})$$

and

$$A^3_{PQ}(x^{n-k}, z^k) = \frac{1}{k-1} \sum_{s=1}^{k-1} A_P(x^{n-s} z^s) A^Q(z^{k-s}, x^{m-k+s}).$$

If $n \leq k$ (resp. $m \leq k$), then A^1_{PQ} (resp. A^2_{PQ}) is equal to zero. By definitions of $\theta(u_k)$ and u_k,

$$\theta(u_k) A^3_{PQ}(x^{n-k}, z^k) = 0$$

for any fixed x. So

$$\partial_{(k)}(u_k)(PQ)(x) = \partial_{(k)}(u_k)(P)(x) Q(x) + P(x) \partial_{(k)}(u_k)(Q)(x).$$

Since $\partial_{(k)}(u_k)$ is linear, it is a differentiation on the algebra $H_b(X)$. The continuity of $\partial_{(k)}(u_k)$ follows from the continuity of $\theta(u_k)$ and the translation T_x.

Let $P \in \mathcal{P}(^n X)$ and $n = km$. From (19) we have that

$$\partial^m_{(k)}(u_k)(P) = \binom{km}{k} \binom{k(m-1)}{k} \cdots \binom{k}{k} \widehat{P}(u_k) = \frac{(mk)!}{(k!)^m} \delta^{(k)}(u_k)(P).$$

Thus

$$\delta^{(k)}(u_k) = \sum_{m=0}^{\infty} \frac{(k!)^m}{(mk)!} \partial^m_{(k)}(u_k).$$

□

This approach can be generalized by the following way. Let $v_p \neq 0$ be an arbitrary element in E_p for some positive integer p. Denote by T_{v_p} the operator on $H_b(X)$

$$T_{v_p}(f) := \widehat{f}(\cdot + v_p).$$

We can write

$$\partial_{(k,p)}(u_k)(\widehat{f})(v_p) := \theta(u_k) \circ T_{v_p}(f).$$

Repeating arguments of Theorem 23, we can see that for every $P \in \mathcal{P}(^{km}X)$,

$$\partial_{(k,k)}(u_k)(\widehat{P})(v_k) = m\widehat{A_P}(v_k^{m-1}, u_k).$$

Moreover, if $f = \sum f_n \in H_b(X)$, then

$$\widehat{f}(v_k + u_k) = \sum_{m=0}^{\infty} \frac{\partial^m_{(k,k)}(u_k)(\widehat{f_{km}})(v_k)}{m!}$$

Aron, Cole and Gamelin in [4] considered the operation $\partial_{(k)}(u_k)$ for the case when $k = 1$ and so $u_k = u_1 = z$ for some $z \in X''$. They used notation $(z)T_x f = (*z)f(x)$ instead $\partial_{(1)}(z)f(x)$. For this special case and using this notation formula (20) can be rewritten as

$$\delta^{(1)}(z)f = \widetilde{\delta}(z)f = \sum_{m=}^{\infty} \frac{1}{m!} z^{*m} = \exp(*z).$$

3.4. **Ball Algebras of Analytic Functions.** In this section we consider maximal ideals of uniform algebras of analytic functions on the ball $r\mathcal{B}$ for some $r > 0$, where $\mathcal{B}$ is the unit ball of a Banach space.

We will consider the following algebras: Let $H^\infty(r\mathcal{B})$ be the algebra of bounded analytic functions on $r\mathcal{B}$, $H^\infty_{uc}(r\mathcal{B})$ be the algebra of uniformly continuous analytic functions on $r\mathcal{B}$ and $H^\infty_c(r\mathcal{B})$ be the algebra of bounded analytic functions on $\mathcal{B}$ which are continuous on the closure $\overline{\mathcal{B}}$. It is clear that

$$H_b(X) \subset H^\infty_{uc}(r\mathcal{B}) \subset H^\infty_c(r\mathcal{B}) \subset H^\infty(r\mathcal{B}).$$

Also it is easy to check that $H^\infty_{uc}(r\mathcal{B})$ consists of precisely the uniform limit on $r\mathcal{B}$ of functions in $H_b(X)$. Since the set of $\phi \in M_b$ satisfying

$R(\phi) \le r)$ is the $H_b(X)$-convex hull of rB in M_b, we have the following theorem.

24. *For each fixed $r > 0$, the compact set $\{\phi \in M_b \colon R(\phi) \le r\}$ coincides with the spectrum of $H^\infty_{uc}(r\mathcal{B})$.*

23. *The spectrum of $H^\infty_{uc}(\mathcal{B})$ contains unit balls of E_k for every k.*

Let now H be a uniform algebra such that $H^\infty_{uc}(r\mathcal{B}) \subset H \subset H^\infty(r\mathcal{B})$ and M_H be its spectrum. There is a natural projection $\iota\colon M_H \to M_b$ such that $\iota(\psi)$ is the restriction of $\psi \in M_H$ to $H_b(X)$. Note that we can extend the definition of the radius function R to $\psi \in M_H$ by declaring $R(\psi)$ to be the smallest value of r, $0 \le r \le 1$, such that ψ is continuous with respect to the norm of uniform convergence on $r\mathcal{B}$.

25. *Let H be a uniform algebra between $H^\infty_{uc}(\mathcal{B})$ and $H^\infty(\mathcal{B})$. The image $\iota(M_H)$ of the projection ι consists of precisely the set $\phi \in M_b$ such that $R(\phi) \le 1$.*

Доведення. If $\psi \in M_H$ and $|\psi(f)| \le \|f\|_{r\mathcal{B}}$ for all $f \in H$, then this inequality holds in particular for all $h \in H_b(X)$, so that $R(\iota(\psi)) \le R(\psi)$ for all $\psi \in M_H$.

Suppose $\phi \in M_b$ satisfies $R(\phi) < 1$. Then ϕ is continuous on $H_b(X)$ with respect to the norm of uniform convergence on $R(\phi)\mathcal{B}$. Now each $f \in H^\infty(\mathcal{B})$ is a uniform limit on any ball $r\mathcal{B}$, $0 < r < 1$ of the partial sums of its Taylor series. Hence ϕ extends uniquely to f and determine a unique $\psi \in M_H$ with $\iota(\psi) = \phi$ and $R(\psi) < 1$. Clearly $R(\phi) = R(\psi)$.

Suppose $\phi \in M_b$ satisfies $R(\phi) = 1$. Let $\phi = \mathop{\ast}\limits_{k=1}^{\infty} \delta^{(k)}(u_k)$. For $|\xi| < 1$, consider the homomorphism $\phi^\xi := \mathop{\ast}\limits_{k=1}^{\infty} \delta^{(k)}(\xi u_k)$. Since $R(\phi^\xi) = |\xi| < 1$, ϕ^ξ extends to a homomorphism in M_H. If ψ is any cluster point in M_H of the extension of the ϕ^ξ as $\xi \to 1$, $|\xi| < 1$, then $\iota(\psi) = \phi$. Thus the image of ι is precisely $\{\phi \in M_b \colon R(\phi) \le 1\}$. $\square$

Comparing Theorem 25 and Theorem 24 we can see that if $H = H^\infty_{uc}(\mathcal{B})$, then the projection ι is one-to-one.

26. *Let H be a uniform algebra between $H^\infty_{uc}(\mathcal{B})$ and $H^\infty(\mathcal{B})$. Then the natural projection of the spectrum M_H of H onto $\{\phi \in M_b \colon R(\phi) \le 1\}$ is one-to-one if and only if $H = H^\infty_{uc}(\mathcal{B})$.*

Доведення. Suppose $f \in H$ is not uniformly continuous. Then there are $\varepsilon > 0$ and sequences (x_n) and (y_n) in $\mathcal{B}$ such that $\|x_n - y_n\| \to 0$, while

$|f(x_n) - f(y_n)| \geq \varepsilon$ for all n. A subnet x_{n_α} converges in M_b to some ϕ satisfying $R(\phi) \leq 1$. The net y_{n_α} then also converges in M_b to ϕ. Since $|f(x_n) - f(y_n)| \geq \varepsilon$, we see that x_{n_α} and y_{n_α} have cluster points θ and θ' in M_H such that $f(\theta) \neq f(\theta')$. However, θ and θ' both coincide with ϕ on $H^\infty_{uc}(\mathcal{B})$, that is θ and θ' both project onto ϕ. $\square$

We notice that in [4] is proved that if X is an infinite-dimensional Banach space, then $H^\infty_{uc}(\mathcal{B}) \neq H^\infty_c(\mathcal{B})$.

3.5. *C^****-Algebras of Continuous Functions.** For a given complex Banach space X we denote by $X^{\Re}$ a Banach space which coincides with X as a point set but endowed with the real structure. In the other words, $X^{\Re}$ is X where we allow real scalar multiplication only. Evidently $X = X^{\Re}$ as topological spaces and each continuous function f on X is well defined and continuous on $X^{\Re}$. We will denote by $f^{\Re}$ the act of f on $X^{\Re}$.

Definition 9. *A mapping $Q\colon X \to \mathbb{C}$ is called an n-degree $*$-polynomial if*

$$Q^{\Re}\colon X^{\Re} \to \mathbb{C}$$

is a complex-valued polynomial of n degree on the real Banach space $X^{\Re}$.

We denote by $\mathcal{P}^*(X)$ the algebra of all $*$-polynomials on X and by $\mathcal{C}_{\mathcal{P}}(\mathcal{B})$ the completion of $\mathcal{P}^*(X)$ in the uniform topology on the unit ball $\mathcal{B}$ of X. $\mathcal{C}_{\mathcal{P}}(\mathcal{B})$ contains all continuous polynomials on X and all continuous *anti-polynomials* on X, where anti-polynomials are just complex conjugates to polynomials. Let us denote by $\mathcal{C}_a(\mathcal{B})$ a minimal closed subalgebra of $\mathcal{C}_{\mathcal{P}}(\mathcal{B})$ which contains all continuous polynomials on X and all continuous anti-polynomials. Notice that $\mathcal{C}_{\mathcal{P}}(\mathcal{B}) \neq \mathcal{C}_a(\mathcal{B})$ in the general case. For example it is easy to check that a $*$-polynomial Q on ℓ_2,

$$Q\Big(\sum_{n=1}^{\infty} x_n e_n\Big) = \sum_{n=1}^{\infty} x_n \overline{x_n}$$

belongs to $\mathcal{C}_{\mathcal{P}}(\mathcal{B})$ but does not belong to $\mathcal{C}_a(\mathcal{B})$.

27. *The spectrum $M(\mathcal{C}_a(\mathcal{B}))$ of $\mathcal{C}_a(\mathcal{B})$ consists of all characters ϕ of $H^\infty_{uc}(\mathcal{B})$ for which there are nets $(x_\alpha) \subset \mathcal{B}$ such that*

$$\phi(P) = \lim_{\alpha} P(x_a) \qquad \forall P \in \mathcal{P}(X)). \tag{21}$$

Доведення. Let $\phi \in H^\infty_{uc}(\mathcal{B})$ such that (21) holds for some $(x_\alpha) \subset \mathcal{B}$. Then $\phi(\overline{P}) := \overline{\phi(P)}$ is well defined for every $P \in \mathcal{P}(X)$. If Q is in an algebraic span of polynomials and antipolynomials, $|\phi(Q)| \leq \sup_\alpha |Q(x_\alpha)| \leq \|Q\|$. So ϕ can be extended by continuity to a character on $\mathcal{C}_a(\mathcal{B})$.

Let now ϕ be a character on $\mathcal{C}_a(\mathcal{B})$. Since $\mathcal{C}_a(\mathcal{B})$ is a C^*-algebra, $M(\mathcal{C}_a(\mathcal{B}))$ is the Czech-Stone compactification of $\mathcal{B}$ in the Gelfand topology of $\mathcal{C}_a(\mathcal{B})$ on $\mathcal{B}$. Hence $\mathcal{B}$ is dense in $\beta\mathcal{B} = M(\mathcal{C}_a(\mathcal{B}))$, that is, there exists a net $(x_\alpha) \subset \mathcal{B}$ such that $\phi(f) = \lim_\alpha f(x_a)$ for every $f \in \mathcal{C}_a(\mathcal{B})$. So (21) holds. $\square$

By the theorem we can write $M(\mathcal{C}_a(\mathcal{B})) \subset M(H^\infty_{uc}(\mathcal{B}))$. Since $M(H^\infty_{uc}(\mathcal{B})) = \{\phi \in M_b \colon R(\phi) \leq 1\}$, we can apply Theorem 19 and Theorem 20.

24. *Let $\phi \in M(\mathcal{C}_a(\mathcal{B}))$. Then there exists a sequence $(u_k)_{k=1}^\infty$, $u_k \in E_k$ such that* $\sup_k \|u_k\|^{1/k} \leq 1$ *and*

$$\phi(f) = \mathop{\circledast}_{k=1}^{\infty} \delta^{(k)}(u_k)(f) \quad \textit{and} \quad \phi(\overline{f}) = \overline{\mathop{\circledast}_{k=1}^{\infty} \delta^{(k)}(u_k)(f)}$$

for every $f \in H^\infty_{uc}(\mathcal{B})$.

A given positive integer m we denote by Q_m a $*$-polynomial on ℓ_{2m} as

$$Q_m(x) = Q_m\Big(\sum_{n=1}^\infty x_n e_n\Big) = \sum_{n=1}^\infty x_n^m \overline{x_n^m}.$$

Let x_α be a weakly polynomially zero net in ℓ_{2m} with $\|x_\alpha\| = 1$, where α belongs to an index set $\mathfrak{A}$. Let $\mathcal{U}$ be a free ultrafilter on $\mathfrak{A}$. We set

$$\psi(f) = \lim_{\mathcal{U}} f(x_\alpha).$$

It is clear that $\psi(f) = f(0)$ if $f \in \mathcal{C}_a(\mathcal{B})$ but $\psi(Q_m) = 1$. So we can see that $\mathcal{C}_a(\mathcal{B}) \neq \mathcal{C}_{\mathcal{P}}(\mathcal{B})$ in ℓ_{2m} and there exists a character ψ in $M(\mathcal{C}_{\mathcal{P}}(\mathcal{B}))$ which vanishes on homogeneous polynomials of $\mathcal{C}_a(\mathcal{B})$.

[1] R. Alencar, R. Aron, P. Galindo, and A. Zagorodnyuk, *Algebra of symmetric holomorphic functions on ℓ_p*, Bull. Lond. Math. Soc. **35** (2003), 55–64.

[2] E.L. Arenson, *Gleason parts and the Choquet boundary of a function algebra on a convex compactum*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 204–207.

[3] R.M. Aron and P.D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), 3–24.

[4] R.M. Aron, B.J. Cole, and T.W. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. Reine Angew. Math. **415** (1991), 51–93.

[5] R.M. Aron, B.J. Cole, and T.W. Gamelin, *Weak-star continuous analytic funtions*, Can. J. Math. **47** (1995), 673–683.

[6] R.M. Aron, C. Hervés, and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. **52** (1983), 189–204.

[7] R.M. Aron, P. Galindo, D. Garcia, and M. Maestre, *Regularity and algebras of analytic function in infinite dimensions*, Trans. Amer. Math. Soc. **348** (1996), 543–559.

[8] R.M. Aron and J.B. Prolla, *Polynomial approximation of differentiable functions on Banach spaces*, J. Reine Angew. Math. **313** (1980), 195–216.

[9] N. Aronzajn, *Teory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[10] Yu. M. Berezanski and Yu. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Kluwer Acad. Publ., Dordrecht, 1995.

[11] E. Bishop, K. De Leeuw, *The representations of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier **9** (1959), 305–331.

[12] P. Biström, J.A. Jaramillo, and M. Lindström, *Polynomial compactness in Banach spaces*, Rocky Mont. J. Math. **28** (1998), 1203–1225.

[13] J. Bonet and A. Peris. *Hypercyclic operators on non-normable Fréchet space//* J. Funct. Anal. – 1998. – **159**. – P. 387–395.

[14] F. Cabello Sánchez, R. García, and I. Villanueva, *Extension of multilinear operators on Banach spaces*, Extracta Math. **15** (2000), 291–334.

[15] D. Carando, D. García and M. Maestre, *Homomorphisms and composition operators on algebras of analytic functions of bounded type*, Adv. Math. To appear.

[16] S.B. Chae, *Holomorphy and Calculus in Normed Spaces,* Pure and Applied Mathematics, A Series of Monographs and Textbooks, vol. 92, Marcel Dekker, Inc., New York, Basel, 1985.

[17] A.M. Davie and T.W. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc. **106** (1989), 351–356.

[18] V. Dimant and R. Gonzalo, *Block diagonal polynomials*, Trans. Amer. Math. Soc. **353** (2000), 733–747.

[19] S. Dineen, *Complex Analysis in Locally Convex Spaces*, Mathematics Studies, vol. 57, North-Holland, Amsterdam, New York, Oxford, 1981.

[20] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Monographs in Mathematics, Springer, New York, 1999.

[21] P. Dixon, *Scalar homomorphisms on algebras of infinetely long polynomials with an application to automatic continuity theory*, J. London Math. Soc. **(2) 19** (1979), 488–496.

[22] N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc. **44** (1938), 305–356.

[23] P. Galindo, D. García, M. Maestre, and J. Mujica, *Extension of multilinear mappings on Banach spaces*, Studia. Math. **108** (1994), 55–76.

[24] Gamelin, T. W., *Uniform algebras,* 2nd ed., Chelsea, New York, 1984.

[25] T.W. Gamelin, *Analytic functions on Banach spaces*, in Complex Function Theory, Ed. Gauthier and Sabidussi, Kluwer Academic Publishers, Amsterdam, 1994, 187–223.
[26] R. Gonzalo, *Multilinear forms, subsymmetric polynomials, and spreading models on Banach spaces*, J. Math. Anal. App. **202** (1996), 379–397.
[27] K.-G. Grosse-Erdmann. *Universal families and hypercyclic operators*// Bull. Amer. Math. Soc. (N.S.). – 1999. – **36**. – P. 345–381.
[28] A. Grotendieck, *Sur certains espaces do fonctions holomorphes,* J. Reine Angew. Math. **192** (1953), 35–64.
[29] M. Hervé, *Analyticity in Infiite Dimensional Spaces*, de Gruyter Stud. in Math., vol. 10, Walter de Gruyter, Berlin, New York, 1989.
[30] E. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. vol. 11, Providence, 1952.
[31] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland, Amsterdam, New York, Oxford, 1986.
[32] J. Mujica, *Ideals of holomorphic functions on Tsirelson's space*, Archiv der Mathematik **76** (2001), 292–298.
[33] C. J. Read. *The invariant subspace problem for a class of Banach spaces. II. Hypercyclic operators*// Israel J. Math. – 1988. – **63**. – P. 1–40.
[34] S. Saitoh, *Integral Transforms, Reproducig Kernels and Their Applications*, Pitman Research Notes in Math. Ser., vol. 369, Longman, 1997.
[35] A. Ülger, *Weakly compact bilinear forms and Arens regularity*, Proc. Amer. Math. Soc. **101** (1987), 697–704.
[36] van der Waerden, B. L., *Modern Algebra*, Ungar (1964).
[37] A. Zagorodnyuk, *Spectra of algebras of entire functions on Banach spaces,* Proc. Amer. Math. Soc. **134** (2006), 2559–2569.

SOME APPLICATIONS OF ELEMENTARY SUBMODELS IN TOPOLOGY

Lyubomyr Zdomskyy

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Wien, Austria.

E-mail address: `lzdomsky@logic.univie.ac.at`

URL: `http://www.logic.univie.ac.at/~lzdomsky/`

Our talks will be devoted to applications of elementary submodels in topology. In particular, we shall present some streamlined proofs of classical results like Arhangel'skiis famous result that the cardinality of first countable compact spaces is at most $\mathfrak{c}$, and some others. We shall also try to present some more recent results like the main combinatorial lemma in the construction of an L-space by J. Moore.

The exposition will mainly follow the article [1].

[1] Dow, A., *An introduction to applications of elementary submodels to topology,* Topology Proc. **13** (1988), 17–72.

ГЕОМЕТРИЧНІ ІНВАРІАНТИ ДИСКРЕТНИХ НЕАВТОНОМНИХ СПРЯЖЕНИХ ЗВОРОТНИХ ДИНАМІЧНИХ СИСТЕМ

Атаманюк О.Б.

Прикарпатський національний університет імені Василя Стефаника, Івано-Франківськ, Шевченка 57, Україна

E-mail address: bogdanatamaniuk@ukr.net

Досліджуються дискретні неавтономні зворотні динамічні системи та їх геометричні властивості [1],[2], які зберігаються при топологічному спряженні. Топологічне спряження - це такий гомеоморфізм між двома динамічними системами (X, T) та (Y, S), для якого виконується рівність: $S \circ \pi = \pi \circ T$.

Теорема 1. *Інваріантом топологічної спряженості є TGE - тонка гомотопічна еквівалентність.*

Теорема 2. *Інваріантом топологічної спряженості є SCU - сильна C-універсальність відображень.*

Теорема 3. *При топологічному спряженні зберігаються майже гомеоморфізми.*

Теорема 4. *При топологічному спряженні зберігається напівнеперервність знизу та напівнеперервність зверху.*

Теорема 5. *При топологічному спряженні зберігається властивість SDAP - сильна дискретна апроксимаційна властивість. За означенням, простір X задовольняє умову SDAP тоді, коли для будь-якого відображення $f : Q \times N \to X$ та для будь-якого покриття $\omega \in Cov(X)$ існує відображення $g : Q \times N \to X$, яке задовольняє дві умови: 1) $(g, f) < \omega$, тобто ω -близькість, 2) сімейство $\{g(Q \times \{n\}) : n \in N\}$ є дискретним сімейством в X.*

Теорема 6. *При топологічному спряженні зберігається властивість C-оборотності та спектральної рухомості.*

Теорема 7. *При топологічному спряженні зберігається DCP - дискова кліткова властивість та DHCP - дискова гомотопічна кліткова властивість.*

Теорема 8. *Для спектрально-рухомих орбіт різновиди м'якості (апроксимативної м'якості) переносяться із зв'язуючих проекцій на граничні проекції (орбіти).*

Теорема 9. *При топологічному спряженні зберігаются різновиди м'якості (апроксимативної м'якості) спектрально-рухомих орбіт дискретних неавтономних зворотних динамічних систем.*

Теорема 10. *Для спектрально-рухомих орбіт зберігається властивість SCU- сильної C- універсальності проекцій при переході від зв'язуючих проекцій до граничних проекцій (орбіт).*

Теорема 11. *При топологічному спряженні спектрально-рухомих орбіт зберігається властивість SCU - сильно C - універсальних орбіт дискретних неавтономних зворотних динамічних систем.*

Література

[1] *Коляда С.Ф.* Топологічна динаміка: мінімальність, ентропія та хаос. – Дисертація доктора фіз.-мат. наук. Київ. (2004), 339с.

[2] *Федорчук В.В., Филиппов В.В.* Общая топология. Основные конструкции: Учеб.пособие. - М.:Изд-во МГУ, (1988), - 252 с.

ЕКСПАНДЕРИ. ІСНУВАННЯ І ПОБУДОВА

О.Д. Глухов

Національний авіаційний університет, Київ, Україна,
E-mail address: a$_{g}$*lukhov@ukr.net*

1. Означення.
2. Спектр графа і алгебрична зв'язність.
3. Випадкові графи. Існування експандерів.
4. Побудова експандерів. Зигзаг-добуток. Графи Рамануджана.
5. Лема про групи перестановок. Перестановочна склейка графів.
6. Експандери із заданими підграфами.
7. Один приклад побудови експандера.

ПРОСТОРИ ЄМНОСТЕЙ НА МЕТРИЧНИХ НЕКОМПАКТНИХ ПРОСТОРАХ

І.Д. Глушак

Прикарпатський національний університет імені Василя Стефаника, м. Івано-Франківськ, вул. Шевченка,57,
E-mail address: `inna-gl@rambler.ru`

Нехай X-метричний некомпактний простір.

Функція $c : \exp X \cup \{\emptyset\} \to I$ називається τ-гладкою ємністю на X, якщо:

1) $c(\emptyset) = 0$, $c(X) = 1$;

2) вона монотонна;

3) для кожної монотонно спадної системи (F_α) замкнених в X множин та множини $G \underset{\mathrm{cl}}{\subset} X$, такої що $\bigcap\limits_\alpha F_\alpha \subset G$, виконується нерівність

$$\inf_\alpha c(F_\alpha) \le c(G).$$

На множині τ-гладких ємностей $\check{M}X$ порівнюються дві топології τ_1 та τ_2. Топологія τ_1 визначена передбазою, яка складаєтьяс з множин вигляду

$$O_-(F,a) = \{c \in \check{M}X \mid c(F) < a\},$$

$$O_+(U,a) = \{c \in \check{M}X \mid \text{існує множина} \mathrm{G} \underset{\mathrm{cl}}{\subset} X,$$
$$\mathrm{G}-\text{цілком відокремлена від } X \setminus U, c(G) > a\},$$

для всіх $F \underset{\mathrm{cl}}{\subset} X$, $U \underset{\mathrm{op}}{\subset} X$, $a \in \mathbb{I}$.

А топологія τ_2 породжена метрикою:

$$\hat{d}(c,c') = \inf\{\delta > 0 \mid c(\bar{O}_\delta(F)) + \delta \geq c'(F), c'(\bar{O}_\delta(F)) + \delta \geq c(F), \forall F \underset{\mathrm{cl}}{\subset} X\}.$$

Література

[1] Zarichnyi M.M., Nykyforchyn O.R., *Functor of capacities in the category of compacta*, Matem. sb., 2008, 199:2, 3-26 (in Russian)

[2] Oleh Nykyforchyn, Dušan Repovs, *Inclusion hyperspaces and capacities on tychonoff spaces: functors and monads*

ПРОЦЕС ГЛОБАЛЬНОЇ ЛІНЕАРИЗАЦІЇ ДЛЯ ДЕЯКИХ ВИДІВ ДРОБОВО-ЛІНІЙНИХ ВІДОБРАЖЕНЬ.

А.В. Загороднюк and М.В. Дубей

Факультет математики та інформатики, Прикарпатський національний університет імені Василя Стефаника, Івано-Франківськ, вул. Шевченка, 57, Україна

E-mail address: andriyzag@yahoo.com

E-mail address: mariadubey@gmail.com

Розглянемо аналітичну функцію

$$\xi(z) = \sum_{n=0}^{\infty} c_n z^n$$

в околі нуля, радіуса $\rho_0(\xi) = \frac{1}{\limsup |c_n|^{1/n}}$.

Нехай X– комплексний банахів простір, $\bigotimes_{\gamma,s}^n X$– n–тий симетричний тензорний степінь простору X, поповнений відносно тензорної норми γ, $\mathcal{F}_{\alpha,\gamma} = \bigoplus\limits_{n=0}^{\infty}(\bigotimes_{\gamma,s}^n X)^{\alpha}$– простір скінченних прямих сум, поповнений відносно деякої норми α, $\Phi_\xi = \{\varphi \circ F_\xi : \varphi \in \mathcal{F}'_{\alpha,\gamma}\}$– клас аналітичних функцій обмеженого типу в кулі $B_{\rho_0(\xi)}$, де φ– неперервний лінійний функціонал на просторі $\mathcal{F}_{\alpha,\gamma}$, а через $F_\xi(x)$ позначимо формальний ряд $\sum\limits_{n=0}^{\infty} c_n x^{\otimes n}$. Тоді при фіксованих ξ, α, γ пара $F_\xi, \mathcal{F}_{\alpha,\gamma}$ задає лінеаризацію функцій з класу Φ_ξ на $B_{\rho_0(\xi)}$. Аналогічно, якщо A– лінійний оператор з $\mathcal{F}_{\alpha,\gamma}$ в деякий нормований простір Y, то $A \circ F_\xi$ буде аналітичним відображенням з $B_{\rho_0(\xi)}$ в Y. У доповіді розглядатимуться відображення вигляду $\xi(z) = \frac{az+b}{cz-d}$, $\xi(z) = \frac{az+b}{-cz+d}$, $\xi(z) = \frac{1}{1-z}$ та процес глобальної лінеаризації цих відображень.

ІНДЕКСИ ДЕЯКИХ ЗЛІЧЕННИХ ГРАФІВ

Андрій Коротков

Київський національний університет імені Тараса Шевченка, Київ, Україна,
E-mail address: myolymp@ukr.net

ОПЕРАДИ ТА ГОМОТОПІЧНІ АЛГЕБРИ

В.В. Любашенко

Інститут математики НАН України, вул. Терещенківська, 3, Київ-4, 01601 МСП
E-mail address: lub@imath.kiev.ua

1. Диференціально–градуйований світ — лінеаризація гомотопічного світу.
2. Операди. Алгебри над операдами.
3. Гомотопічні алгебри — алгебри над **dg**-резольвентами стандартних операд.
4. Гомотопічно асоціативні алгебри (A_∞-алгебри).
5. Морфізми A_∞-алгебр утворюють бімодуль над операдою A_∞.
6. Гомотопічно унітальні A_∞-алгебри.
7. Морфізми гомотопічно унітальних A_∞-алгебр як бімодуль над операдою.
8. Мультикатегорії — кольорові операди.
9. Морфізми A_∞-алгебр з кількома аргументами.
10. Розслаблені моноїдальні $\mathcal{C}at$-категорії.
11. $\mathcal{C}at$-двосхили та $\mathcal{C}at$-мультикатегорії.
12. Розслаблені $\mathcal{C}at$-операди та $\mathcal{C}at$-мультикатегорії.
13. Розслаблена $\mathcal{C}at$-операда DG.
14. Модуль над операдою з $n+1$ дією ($n \wedge 1$-модуль).
15. A_∞-морфізми з n аргументами утворюють $n \wedge 1$-модуль над A_∞.
16. Гомотопічно унітальні A_∞-морфізми з n аргументами.

ГОМОМОРФІЗМИ АЛГЕБРИ $\mathcal{P}_{vs}(\mathcal{X}^2_\infty)$.

А.В. Загороднюк and В. В. Максимів

Факультет математики та інформатики, Прикарпатський національний університет імені Василя Стефаника, Івано-Франківськ, вул. Шевченка, 57, Укаїна

E-mail address: andriyzag@yahoo.com

E-mail address: maksymivvika@gmail.com

Нехай X — банахів простір із симетричним базисом. Очевидно, що X можна розглядати, як простір числових послідовностей. Позначимо $\mathcal{P}_s(X)$ алгебру поліномів на X, які є симетричними (інваріантними) відносно перестановок елементів цих послідовностей.

У цій роботі ми досліджуємо поліноми на декартових добутках банахових просторів із симетричним базисом, які є інваріантними відносно дії деякої природної підгрупи $\mathcal{S}(\mathbb{N})$ (ми будемо їх називати блочно-симетричними). Точніше, нехай: $\mathcal{X}_\infty^\infty = \left(\sum X\right)_{l_1} = \oplus_{l_1} X$. Тоді кожен елемент $\overline{x} \in \mathcal{X}_\infty^\infty$ можна подати у вигляді послідовності $\overline{x} = (x_1, \ldots, x_n, \ldots)$, де $x_n \in X$ з нормою $\|\overline{x}\| = \sum\limits_{k=1}^{\infty} \|x_k\|$. Будемо казати, що поліном P на просторі $\mathcal{X}_\infty^\infty$ називається *блочно-симетричним (векторно-симетричним)*, якщо: $P(x_1, , \ldots, x_n, \ldots) = P(x_{\sigma(1)}, , \ldots, x_{\sigma(n)}, \ldots)$ для будь-якої блочної перестановки σ. Позначимо через $\mathcal{P}_{vs}(\mathcal{X}_\infty^\infty)$ алгебру блочно-симетричних поліномів на просторі $\mathcal{X}_\infty^\infty$.

Справедливим є твердження: ***Нехай*** $\mathcal{X}_m^n = \oplus_1^m \mathbb{C}^n$. ***Тоді*** $\mathcal{P}_{vs}(\mathcal{X}_m^n)$ ***має скінченну систему твірних.***

У доповіді буде описано твірні елементи $\mathcal{P}_{vs}(\mathcal{X}_\infty^\infty)$ у двох випадках: $\mathcal{X}_2^n = \mathbb{C}^n \oplus \mathbb{C}^n$ і $\mathcal{X}_m^2 = \oplus_1^m \mathbb{C}^2$. Також буде показано, що існує неперервний гомоморфізм з алгебри $\mathcal{P}_{vs}(\mathcal{X}_\infty^2)$ у алгебру $\mathcal{P}_s(l_1)$, який є проектором і неперервний гомоморфізм з алгебри $\mathcal{P}_{vs}(\mathcal{X}_\infty^2)$ у алгебру $\mathcal{P}(l_1)$.

Література

[1] Чернега І. В. *Симетричні поліноми на банахових просторах*, Карп. мат. публікації – 2009. – Т.1 №2. – с. 105-125.

[2] Alencar R., Aron R., Galindo P., Zagorodnyuk A. *Algebra of symmetric holomorphic functions on ℓ_p*, Bull. Lond. Math. Soc., **35** (2003), 55-64.

[3] Gonzalez M., Gonzalo R. and Jaramillo J. *Symmetric polynomials on rearrangement invariant function spaces*, Jour. London Math. Soc. – 1999. – Vol. 59. – P. 681–697.

ТЕОРІЯ МОРСА ТА ЇЇ ЗАСТОСУВАННЯ

О.О.Пришляк

Київський національний університет імені Тараса Шевченка, Київ, Україна,
E-mail address: prishlyak@yahoo.com

ЛОКАЛЬНІ МАЙЖЕ-КІЛЬЦЯ ПОРЯДКУ p^3 З НЕАБЕЛЕВОЮ АДИТИВНОЮ ГРУПОЮ ЕКСПОНЕНТИ p

І.Ю. Раєвська

Інститут математики НАН України, Київ, вул. Терещенківська, 3, Україна
E-mail address: raemarina@rambler.ru

Алгебраїчна структура R з двома бінарними операціями $+$ і $\cdot$ називається (лівим) *майже-кільцем*,якщо $(R,+)$ – необов'язково абелева група, $(R,\cdot)$ – напівгрупа та $r(s+t) = rs+rt$ для всіх $r, s, t \in R$. Група $(R,+)$ позначається через R^+ та називається *адитивною групою*, а її нейтральний елемент 0 – *нулем* майже-кільця R. Очевидно $r \cdot 0 = 0$ для кожного $r \in R$. Майже-кільце R називається *нуль-симетричним*, якщо $0 \cdot r = 0$ та *майже-кільцем з одиницею*, якщо напівгрупа $(R,\cdot)$ є моноїдом. Група всіх оборотних елементів моноїда $(R,\cdot)$ називається *мультиплікативною групою* в R та позначається через R^*. Майже-кільце R з одиницею називається *локальним*, якщо множина L_R всіх необоротних елементів із $(R,\cdot)$ утворює адитивну підгрупу в R^+, і *майже-полем*, якщо $L_R = 0$.

Локальні майже-кільця із скінченною абелевою адитивною $p-$групою вивчалися у роботі. В описані всі неізоморфні нуль-симетричні локальні майже-кільця з елементарною абелевою адитивною групою порядку p^2, які не є майже-полями. В даній роботі наводяться необхідні та деякі достатні умови існування локальних майже-кілець на неабелевій адитивній групі порядку p^3 та експоненти p. Як відомо, для таких груп $p > 2$, а комутант співпадає з центром і має порядок p.

Нехай $R-$ локальне майже-кільце, адитивна група R^+ якого неабелева порядку p^3 та експоненти p, та L – множина всіх необоротних елементів із R. Тоді L – нормальна підгрупа порядку p^2 в R^+ і, отже, $R^+ =< e_1 > +L$, де e_1 – одиничний елемент в R. Оскільки L містить комутант групи R^+, то її твірні e_2 та e_3 можна вибрати так, що $e_3 = -e_1 - e_2 + e_1 + e_2$. Тоді $L =< e_2 > + < e_3 >$ і підгрупа $< e_3 >$ є центром групи R^+. Отже, якщо $r \in R$, то $r = e_1r_1 + e_2r_2 + e_3r_3$ з коефіцієнтами r_1, r_2, r_3, які можна розглядати як елементи поля F_p лишків по модулю p, що однозначно визначаються елементом r. Таким чином, для кожного

$x \in R$ та кожного $i \in \{1,2,3\}$ однозначно визначені елементи $\rho_{1j}(x), \rho_{2j}(x), \rho_{3j}(x)$ поля F_p, а отже відображення $\rho_{ij} : R \to F_p$, для яких $xe_j = e_1\rho_{1j}(x)+e_2\rho_{2j}(x)+e_3\rho_{3j}(x)$. Очевидно, що $\rho_{i1}(x) = x_i$ для $i \in \{1,2,3\}$, оскільки $xe_1 = x$ для кожного $x \in R$.

Лема 1. *Для кожного $x = x_1e_1 + x_2e_2 + x_3e_3 \in R$ виконуються рівності $\rho_{12}(x) = \rho_{13}(x) = \rho_{23}(x) = 0$ та $\rho_{33}(x) = x_1\rho_{22}(x)$.*

Лема 2. *Якщо $x, y \in R$, то*

$$xy = (xe_1)y_1 + (xe_2)y_2 + (xe_3)y_3 = e_1(x_1y_1) + e_2(x_2y_1 + \rho_{22}(x)y_2)+$$
$$e_3(x_3y_1 + \rho_{23}(x)y_2 + x_1\rho_{22}(x)y_3 + x_1x_2 \begin{pmatrix} y_1 \\ 2 \end{pmatrix}),$$

причому відображення $\rho_{22} : R \to F_p$ та $\rho_{23} : R \to F_p$ задовольняють умовам:

(1) $\rho_{22}(xy) = \rho_{22}(x)\rho_{22}(y)$,
(2) $\rho_{23}(xy) = \rho_{23}(x)\rho_{22}(y) + x_1\rho_{22}(x)\rho_{23}(y)$.

Теорема 1. *Кожне локальне майже-кільце R з неабелевою адитивною групою порядку p^3 та експоненти p визначається відображеннями $\rho_{22} : R \to F_p$ та $\rho_{23} : R \to F_p$, що задовольняють умовам (1) та (2) леми 2. Більш того, майже-кільце R нуль-симетричне тоді і тільки тоді, коли $\rho_{22}(0) = 0$.*

Навпаки, нехай G – адитивна неабелева група порядку p^3 та експоненти p з твірними e_1, e_2 та $e_3 = -e_1 - e_2 + e_1 + e_2$. Тоді $G = < e_1 > + < e_2 > + < e_3 >$ і кожний елемент $x \in G$ однозначно записується у вигляді $x = x_1e_1 + x_2e_2 + x_3e_3$, де коефіцієнти x_1, x_2, x_3 можна розглядати як елементи поля F_p.

Теорема 2. *Якщо відображення $\rho_{22} : R \to F_p$ та $\rho_{23} : R \to F_p$ задовольняють умовам $\rho_{22}(x) = x_1$ та $\rho_{23}(x) = x_1(1-x_1)$ для кожного $x \in G$, то операція*

$$e_1(x_1y_1)+e_2(x_2y_1+\rho_{22}(x)y_2)+e_3(x_3y_1+\rho_{23}(x)y_2+x_1\rho_{22}(x)y_3+x_1x_2 \begin{pmatrix} y_1 \\ 2 \end{pmatrix})$$

на адитивній групі G є асоціативною та ліво-дистрибутивною і визначає деяке нуль-симетричне локальне майже-кільце $R = (G, +, \cdot)$.

Література

[1] Carlton J. Maxson, *Local near-rings of cardinality* p^2, Canad. Math. Bull. vol. **11**(4) (1968).
[2] Carlton J. Maxson, *On the construction of finite local near-rings (I): on noncyclic abelian p-groups*, Quart. J. Math. Oxford (2), 21 (170), 449–57.

ПРО МУЛЬТИПЛІКАТИВНІ ГРУПИ МАЙЖЕ-ПОЛІВ

М. Ю. Раєвська

Інститут математики НАН України, Київ, вул. Терещенківська, 3, Україна
E-mail address: raemarina@rambler.ru

В роботі [1] мультиплікативна група майже-поля названа *спадковою*, якщо кожна її підгрупа ізоморфна мультиплікативній групі деякого майже-поля, та наведена повна класифікація таких груп. Нижче розглядаються мультиплікативні групи майже-полів, в яких лише неабелеві підгрупи задовольняють даній умові. Нагадаємо, що *майже-полем* називається алгебраїчна структура F з двома операціями, додаванням та множенням, що задовольняє наступним умовам:

(1) F утворює групу F^+ відносно додавання, яка називається *адитивною групою* майже-поля F;
(2) множина ненульових елементів $F^* = F \setminus 0$ із F утворює групу відносно множення, яка називається *мультиплікативною групою* майже-поля F;
(3) в F виконується односторонній (наприклад, лівий) дистрибутивний закон, тобто $a(b+c) = ab + ac$ для всіх $a, b, c \in F$.

Скінченні майже-поля вивчались Цассенхаузом в [2] (див. також [3], теорема 20.7.2). Зокрема, ним було встановлено, що їх адитивні группи є елементарними абелевими, та детально описано будову мультиплікатив
групп таких майже-полів.

Нами доведена наступна теорема.

Теорема 1. *Нехай F - скінченне майже-поле, кожна неабелева підгрупа мультиплікативної групи F^* якого ізоморфна мультиплікативній групі деякого майже-поля. Тоді F^* - група одного з наступних типів:*

(1) *циклічна група;*
(2) *група кватерніонів Q_8;*
(3) *неабелева метациклічна група порядку 24;*
(4) *спеціальна лінійна группа $SL(2,3)$ степеня 2 над полем із 3-х елементів;*
(5) *неабелева метациклічна група порядку 63;*
(6) *неабелева метациклічна група порядку 80.*

Література

[1] Ligh S. *Finite Hereditary Near-field groups*, Mh. Math. 86 (1978), 7-11.
[2] Zassenhaus H. *"Uber endliche Fastkörper*, Ab. Math. Sem. Univ. Hamburg, 11 (1935/36), 187-220.
[3] Холл М. *Теория групп*, М.: Издательство иностранной литературы, 1962, 468с.

Participants

Учасники

Contents

Зміст