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SEMIGROUPS OF CENTERED UPFAMILIES ON FINITE MONOGENIC SEMIGROUPS

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Abstract

Given a finite monogenic semigroup S, we study the minimal ideal, the center, left cancelable, and right cancelable elements of the extension $N_{<\omega}(S)$ consisting of centered upfamilies on S and characterize monogenic semigroups whose extensions are commutative.

1. Introduction

This paper is devoted to describing the structure of extensions $N_{<\omega}(S)$ of monogenic semigroups S. The thorough study of various extensions of semigroups was started in [11] and continued in [1]-[8], [12]-[15]. The largest among these extensions is the semigroup $\upsilon(S)$ of all upfamilies on S. A family $\mathcal{M}$ of nonempty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{M}$ any subset $B \supset A$ of X belongs to $\mathcal{M}$. Each family $\mathcal{B}$ of nonempty subsets of X generates the upfamily

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$\langle B \subset X : B \in \mathcal{B} \rangle = \{A \subset X : \exists B \in \mathcal{B}(B \subset A)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called the *Stone-Čech compactification* of X (see [16], [19]). An ultrafilter $\langle\{x\}\rangle$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Identifying each point $x \in X$ with the principal ultrafilter $\langle\{x\}\rangle$ we obtain the inclusions $X \subset \beta(X) \subset \upsilon(X)$. It was shown in [11] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $\circ : \upsilon(S) \times \upsilon(S) \to \upsilon(S)$ by the formula

$$\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle,$$

for upfamilies $\mathcal{L}, \mathcal{M} \in \upsilon(S)$. In this case, the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $\upsilon(S)$. The semigroup $\upsilon(S)$ contains many other important extensions of S. In particular, it contains the semigroup $N_{<\omega}(S)$ of centered upfamilies. An upfamily $\mathcal{L} \in \upsilon(S)$ is called *centered* if $\bigcap \mathcal{F} \neq \emptyset$ for any finite subfamily $\mathcal{F} \subset \mathcal{L}$.

Each map $f : X \to Y$ induces the map

$$N_{<\omega} f : N_{<\omega}(X) \to N_{<\omega}(Y), \quad N_{<\omega} f : \mathcal{M} \mapsto \langle f(M) \subset Y : M \in \mathcal{M} \rangle,$$

see [10].

A nonempty subset I of a semigroup $(S, *)$ is called an *ideal* (resp., a *right ideal*, a *left ideal*) if $I * S \cup S * I \subset I$ (resp., $I * S \subset I$, $S * I \subset I$). An element z of a semigroup $(S, *)$ is called a *zero* (resp., a *left zero*, a *right zero*) in S if $a * z = z * a = z$ (resp., $z * a = z$, $a * z = z$) for any $a \in S$. It is clear that $z \in S$ is a zero (resp., a left zero, a right zero) in S if and only if the singleton $\{z\}$ is an ideal (resp., a right ideal, a left ideal) in S. An ideal $I \subset S$ is called *minimal* if any ideal of S that lies in I coincides with I. By analogy, we define minimal left and minimal right ideals of S. The union $K(S)$ of all minimal left (right) ideals of S coincides with the minimal ideal of S, see [16, Theorem 2.8]. A semigroup $(S, *)$ is said to be a *right zero semigroup* if $a * b = b$ for any $a, b \in S$. A map $\varphi : S \to T$ between semigroups $(S, *)$ and $(T, \circ)$ is called a *homomorphism* if $\varphi(a * b) = \varphi(a) \circ \varphi(b)$ for any $a, b \in S$. A homomorphism $\varphi : S \to I$ from a semigroup S onto an ideal $I \subset S$ is called a *retraction* if $\varphi(a) = a$ for any element $a \in I$. An element a of a semigroup S is called *left cancelable* (resp., *right cancelable*) if for any elements $x, y \in S$ the equality $ax = ay$ (resp., $xa = ya$) implies $x = y$. This is equivalent to saying that the left (resp., right) shift $l_a : S \to S$, $l_a : x \mapsto a * x$ (resp., $r_a : S \to S$, $r_a : x \mapsto x * a$) is injective. A semigroup S is called *left* (*right*) *cancellative* if all elements of S are left (right) cancelable. A semigroup that is both left and right cancellative is said to be *cancellative*. By definition, the *center* of a semigroup S is the set $C(S) = \{a \in S : \forall s \in S \ (sa = as)\}$.

A semigroup $\langle a \rangle = \{a^n\}_{n \in \mathbb{N}}$ generated by a single element a is called *monogenic* or *cyclic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$. A finite monogenic semigroup $S = \langle a \rangle$ also has very simple structure (see [9]). There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \ldots, a^{m+r-1}\}$ and $m + r - 1 = |S|$;
- for any $i, j \in \omega$ the equality $a^{r+i} = a^{r+j}$ holds if and only if $i \equiv j \mod m$;
- $C_m = \{a^r, a^{r+1}, \ldots, a^{m+r-1}\}$ is the minimal ideal, a cyclic and maximal subgroup of S with the neutral element $e = a^n \in C_m$, where m divides n.

From now on we denote by $C_{r,m}$ a finite monogenic semigroup of index r and period m, and maximal subgroup of $C_{r,m}$ is denoted by C_m.

2. Homomorphisms, Zeros and Minimal Ideals

Proposition 2.1. *For any homomorphism* $\varphi : S \to T$ *between semigroups* $(S, *_1)$ *and* $(T, *_2)$ *the induced map* $N_{<\omega}\varphi : N_{<\omega}(S) \to N_{<\omega}(T)$ *is a homomorphism of the semigroups* $(N_{<\omega}(S), \circ_1)$ *and* $(N_{<\omega}(T), \circ_2)$.

Proof. Given two centered upfamilies $\mathcal{L}, \mathcal{M} \in N_{<\omega}(S)$ observe that

$$N_{<\omega}\varphi(\mathcal{L} \circ_1 \mathcal{M}) = N_{<\omega}\varphi(\langle \bigcup_{x \in L} x *_1 M_x : L \in \mathcal{L}, \{M_x\}_{x \in L} \subset \mathcal{M} \rangle)$$

$$= \langle \varphi(\bigcup_{x \in L} x *_1 M_x) : L \in \mathcal{L}, \{M_x\}_{x \in L} \subset \mathcal{M} \rangle$$

$$= \langle \bigcup_{x \in L} \varphi(x) *_2 \varphi(M_x) : L \in \mathcal{L}, \{M_x\}_{x \in L} \subset \mathcal{M} \rangle$$

$$= \langle \bigcup_{x \in \varphi(L)} x *_2 \varphi(M_x) : L \in \mathcal{L}, \{\varphi(M_x)\}_{x \in \varphi(L)} \subset N_{<\omega}\varphi(\mathcal{M}) \rangle$$

$$= \langle \varphi(L) : L \in \mathcal{L} \rangle \circ_2 \langle \varphi(M) : M \in \mathcal{M} \rangle = N_{<\omega}\varphi(\mathcal{L}) \circ_2 N_{<\omega}\varphi(\mathcal{M}).$$

□

Let us note that for a subsemigroup T of a semigroup S the homomorphism $i : N_{<\omega}(T) \to N_{<\omega}(S), i : \mathcal{A} \to \langle \mathcal{A} \rangle_S$ is injective, and thus we can identify the semigroup $N_{<\omega}(T)$ with the subsemigroup $i(N_{<\omega}(T)) \subset N_{<\omega}(S)$. Therefore, for each family $\mathcal{B}$ of nonempty subsets of T, we identify the upfamilies

$$\langle \mathcal{B} \rangle_T = \{A \in T \mid \exists B \in \mathcal{B}(B \subset A)\} \in N_{<\omega}(T),$$

and

$$\langle \mathcal{B} \rangle_S = \{A \in S \mid \exists B \in \mathcal{B}(B \subset A)\} \in N_{<\omega}(S).$$

Lemma 2.2. *Let I be an ideal of a semigroup S. If a map* $\varphi : S \to I$ *is a retraction, then the map* $N_{<\omega}\varphi : N_{<\omega}(S) \to N_{<\omega}(I)$ *is a retraction too.*

Proof. Indeed, let $\mathcal{A} \in N_{<\omega}(I), \mathcal{M} \in N_{<\omega}(S)$, then $\mathcal{A} \circ \mathcal{M} = \left\langle \bigcup_{a\in A} a * M_a : A \in \mathcal{A},\, A \subset I,\, \{M_a\}_{a\in A} \subset \mathcal{M} \right\rangle = \left\langle \bigcup_{a\in A} a * M_a : A \in \mathcal{A}, \{M_a\}_{a\in A} \subset \mathcal{M}, \bigcup_{a\in A} a * M_a \subset I \right\rangle \in N_{<\omega}(I)$. By analogy $\mathcal{M} \circ \mathcal{A} \in N_{<\omega}(I)$, and therefore $N_{<\omega}(I)$ is an ideal of the semigroup $N_{<\omega}(S)$. If $\mathcal{A} \in N_{<\omega}(I)$, then $N_{<\omega}\varphi(\mathcal{A}) = \langle \varphi(A) : A \subset I, A \in \mathcal{A} \rangle = \langle A : A \subset I, A \in \mathcal{A} \rangle = \{A \in \mathcal{A} : A \subset I\} = \mathcal{A}$ and hence $N_{<\omega}\varphi$ is a retraction. □

Let e be the neutral element of the maximal subgroup C_m of a monogenic semigroup $C_{r,m}$.

Lemma 2.3. *The map* $\varphi : C_{r,m} \to C_m$, $\varphi(x) = ex$ *is a retraction and* $\varphi(x)y = xy$ *for any* $x \in C_{r,m}$ *and* $y \in C_m$.

Proof. Since the semigroup C_m is an ideal of the semigroup $C_{r,m}$, $\varphi(x) = ex \in C_m$. Consequently, $\varphi(xy) = exy = eexy = exey = \varphi(x)\varphi(y)$ for any $x, y \in C_{r,m}$ and $\varphi(x) = ex = x$ for $x \in C_m$. Hence the map $\varphi : C_{r,m} \to C_m$ is a retraction. Further for any $x \in C_{r,m}$ and $y \in C_m$, we have that $xy \in C_m$, and therefore $\varphi(xy) = xy$. On the other hand, $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(x)y$, since $y \in C_m$. □

Proposition 2.4. *For each finite monogenic semigroup* $C_{r,m}$, *the centered upfamily* $\langle C_m \rangle$ *is the zero of the semigroup* $N_{<\omega}(C_{r,m})$.

Proof. Let $\varphi : C_{r,m} \to C_m$ be the retraction from Lemma 2.3. Since $xF \supset xC_m = \varphi(x)C_m = C_m$ for each $x \in C_{r,m}$ and $F \in \langle C_m \rangle$, then $xF \in \langle C_m \rangle$ and $\langle C_m \rangle$ is a right zero according to Proposition 1 from [15].

We shall show that $\langle C_m \rangle$ is a left zero, that is $\langle C_m \rangle \circ \mathcal{A} = \langle C_m \rangle$ for each $\mathcal{A} \in N_{<\omega}(C_{r,m})$. Since $C_m A = C_m \varphi(A) = C_m$ for $A \in \mathcal{A}$, then $\langle C_m \rangle \subset \langle C_m \rangle \circ \mathcal{A}$. If $M \in \langle C_m \rangle \circ \mathcal{A}$, then $M \supset \bigcup_{g \in C_m} gA_g, \{A_g\}_{g \in C_m} \subset \mathcal{A}$. Since $\mathcal{A}$ is a centered upfamily and C_m is finite, then there exists $a \in \bigcap_{g \in C_m} A_g$. Therefore, $M \supset \bigcup_{g \in C_m} g\{a\} = C_m a = C_m \varphi(a) = C_m$ and $M \in \langle C_m \rangle$. □

Since an element z of a semigroup S is a zero in S if and only if the singleton $\{z\}$ is an ideal in S, Proposition 2.4 implies the following:

Proposition 2.5. *The minimal ideal of the semigroup* $N_{<\omega}(C_{r,m})$ *is singleton, that is* $K(N_{<\omega}(C_{r,m})) = \{\langle C_m \rangle\}$.

3. Commutativity and the Center

Theorem 3.1. *A finite monogenic semigroup* $C_{r,m} = \{a, \dots, a^r, \dots, a^{m+r-1} | a^{r+m} = a^r\}$ *of order* $m + r - 1$ *has commutative extension* $N_{<\omega}(C_{r,m})$ *if and only if*

$$(r, m) \in \{(1,1), (1,2), (2,1), (2,2), (3,1)\}.$$

Proof. It was proved in the paper [15] that the extension $N_{<\omega}(G)$ of a group G is commutative if and only if $|G| \leq 2$. Since for $m > 2$, the extension $N_{<\omega}(C_{r,m})$ contains a noncommutative subsemigroup $N_{<\omega}(C_m)$, then $N_{<\omega}(C_{r,m})$ is not commutative. So it is sufficient to consider only finite monogenic semigroups of period $m \leq 2$.

If index $r = 1$, then $C_{r,m}$ is a cyclic group of order m, and thus for $r = 1$, the semigroup $N_{<\omega}(C_{r,m})$ is commutative if and only if $m \leq 2$.

If $r = 2,\ m \in \{1, 2\}$, then the product xy of any two elements of $C_{r,m}$ is contained in the maximal subgroup $xy \in C_m$, and thus $xy = \varphi(xy) = \varphi(x)\varphi(y)$, where $\varphi : C_{r,m} \to C_m$ is the retraction $\varphi : s \to es$. Since extensions $N_{<\omega}(G)$ of groups G of order 1 and 2 are commutative and the map $N_{<\omega}\varphi : N_{<\omega}(C_{r,m}) \to N_{<\omega}(C_m)$ is a retraction according to Proposition 2.2, then

$$\mathcal{A} \circ \mathcal{B} = N_{<\omega}\varphi(\mathcal{A}) \circ N_{<\omega}\varphi(\mathcal{B}) = N_{<\omega}\varphi(\mathcal{B}) \circ N_{<\omega}\varphi(\mathcal{A}) = \mathcal{B} \circ \mathcal{A},$$

for any $\mathcal{A}, \mathcal{B} \in N_{<\omega}(C_{r,m})$. Consequently, the semigroups $N_{<\omega}(C_{2,1})$ and $N_{<\omega}(C_{2,2})$ are commutative.

Consider the semigroup $C_{3,1} = \{a, a^2, a^3 : a^4 = a^3\}$. The semigroup $N_{<\omega}(C_{3,1})$ contains 10 elements.

Let us introduce the notation

$$\big|^x_y = \big|^y_x = \langle\{x, y\}\rangle, \quad \vee_x = \{F \subset C_{3,1} : |F| \geq 2,\ x \in F\}.$$

Also the principal ultrafilter $\langle\{e\}\rangle$ and the upfamily $\{C_{3,1}\}$ are identified with e and $C_{3,1}$, respectively.

The following Cayley table implies the commutativity of $N_{<\omega}(C_{3,1})$.

$\circ$	a	a^2	a^3	$\vert_{a}^{a^2}$	$\vert_{a}^{a^3}$	$\vert_{a^2}^{a^3}$	$\vee_a$	$\vee_{a^2}$	$\vee_{a^3}$	$C_{3,1}$
a	a^2	a^3	a^3	$\vert_{a^2}^{a^3}$	$\vert_{a^2}^{a^3}$	a^3	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$
a^2	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\vert_{a}^{a^2}$	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$	$\vert_{a^2}^{a^3}$	a^3	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$
$\vert_{a}^{a^3}$	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$	$\vert_{a^2}^{a^3}$	a^3	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$
$\vert_{a^2}^{a^3}$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\vee_a$	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$	$\vert_{a^2}^{a^3}$	a^3	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$
$\vee_{a^2}$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$\vee_{a^3}$	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3	a^3
$C_{3,1}$	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$	$\vert_{a^2}^{a^3}$	a^3	$\vert_{a^2}^{a^3}$	a^3	a^3	$\vert_{a^2}^{a^3}$

Consider the semigroup $C_{3,2} = \{a, a^2, a^3, a^4 : a^5 = a^3\}$ and centered upfamilies $\mathcal{A} = \langle\{a, a^2\}\rangle$, $\mathcal{B} = \langle\{a, a^2\}, \{a, a^3\}\rangle$ of the semigroup $N_{<\omega}(C_{3,2})$. Since

$$\mathcal{A} \circ \mathcal{B} = \langle\{a^2, a^3\}\rangle \neq \langle\{a^2, a^3, a^4\}\rangle = \mathcal{B} \circ \mathcal{A},$$

then the semigroup $N_{<\omega}(C_{3,2})$ is not commutative.

Let $r \geq 4$. Consider centered upfamilies $\mathcal{A} = \langle\{a, a^2\}\rangle$ and $\mathcal{B} = \langle\{a, a^2\}, \{a^2, a^3\}\rangle$. We have that $a^3 \notin a\{a, a^3\} \cup a^2\{a^2, a^3\} \in \mathcal{A} \circ \mathcal{B}$, but $a^3 \in F$ for any $F \in \mathcal{B} \circ \mathcal{A}$. Consequently, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{B} \circ \mathcal{A}$ and for each $r \geq 4$ a semigroup $N_{<\omega}(C_{r,m})$ is not commutative. $\square$

Let us study the center of the semigroup $N_{<\omega}(C_{r,m})$. Since monogenic semigroups are commutative, then the formula

$$\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$$

implies that the center $C(N_{<\omega}(C_{r,m}))$ of a semigroup $N_{<\omega}(C_{r,m})$ contains all principal ultrafilters. It was shown in [15] that $C(N_{<\omega}(G)) = \{\langle\{g\}\rangle : g \in G\} \cup \{\langle G\rangle\}$ for a finite Abelian group G, that is the center $C(N_{<\omega}(G))$ of the semigroup $N_{<\omega}(G)$ is isomorphic to G^0. Therefore, $C(N_{<\omega}(C_m)) \cong (C_m)^0$.

Lemma 3.2. *Let* $\varphi : S \to I$ *be a retraction of a semigroup* S *on an ideal* I. *If* $a \in C(S)$, *then* $\varphi(a) \in C(I)$.

Proof. Indeed, for any $x \in I$, we have

$$\varphi(a)x = \varphi(a)\varphi(x) = \varphi(ax) = \varphi(xa) = \varphi(x)\varphi(a) = x\varphi(a).$$

□

Theorem 3.3. *For each finite monogenic semigroup* $C_{r,m}$, *the center of the semigroup* $N_{<\omega}(C_{r,m})$ *contains centered upfamilies that are neither principal ultrafilters nor the zero of* $N_{<\omega}(C_{r,m})$. *The center of the semigroup* $N_{<\omega}(C_{2,m})$ *contains* $m + 5$ *elements.*

Proof. Since by Lemmas 2.3 and 2.2 the maps $\varphi : C_{r,m} \to C_m$, $\varphi(x) = ex$ and $N_{<\omega}\varphi : N_{<\omega}(C_{r,m}) \to N_{<\omega}(C_m)$ are retractions and $\langle C_m \rangle$ is the zero of $N_{<\omega}(C_{r,m})$ and $N_{<\omega}(C_m)$, then Lemma 3.2 implies that

$$C(N_{<\omega}(C_{r,m})) \subset (N_{<\omega}\varphi)^{-1}(\{\langle\{g\}\rangle : g \in C_m\} \cup \{\langle C_m \rangle\}).$$

Let a be a generator of a semigroup $C_{r,m}$. Consider elements a^{r-1} and $\varphi(a^{r-1}) = ea^{r-1} \neq a^{r-1}$. We claim that centered upfamilies $\mathcal{A} = \langle\{a^{r-1}, ea^{r-1}\}\rangle$, $\mathcal{B} = \langle\{a^{r-1}\} \cup C_m\rangle$, and $\mathcal{C} = \langle\{a^{r-1}\} \cup C_m \setminus \{ea^{r-1}\}\rangle$ are central elements of the semigroup $N_{<\omega}(C_{r,m})$.

Indeed, since $a^{r-1}x \in C_m = \{a^r, \dots, a^{r+m-1}\}$ for each $x \in C_{r,m}$, then $a^{r-1}x = \varphi(a^{r-1}x) = \varphi(a^{r-1})\varphi(x)$. On the other hand, since C_m is an ideal of $C_{r,m}$, then $\varphi(a^{r-1})x \in C_m$ and

$$\varphi(a^{r-1})x = \varphi(\varphi(a^{r-1})x) = \varphi(\varphi(a^{r-1}))\varphi(x) = \varphi(a^{r-1})\varphi(x).$$

Consequently, $\varphi(a^{r-1})x = a^{r-1}x$ for any $x \in C_{r,m}$. Therefore,

$$\mathcal{A} \circ \mathcal{M} = \langle\{\varphi(a^{r-1})\}\rangle \circ \mathcal{M} = \mathcal{M} \circ \langle\{\varphi(a^{r-1})\}\rangle = \mathcal{M} \circ \mathcal{A},$$

$$\mathcal{B} \circ \mathcal{M} = \langle C_m\rangle \circ \mathcal{M} = \mathcal{M} \circ \langle C_m\rangle = \mathcal{M} \circ \mathcal{B},$$

$$\mathcal{C} \circ \mathcal{M} = \langle C_m\rangle \circ \mathcal{M} = \mathcal{M} \circ \langle C_m\rangle = \mathcal{M} \circ \mathcal{C},$$

for any $\mathcal{M} \in N_{<\omega}(C_{r,m})$ and thus $\mathcal{A}, \mathcal{B}, \mathcal{C} \in C(N_{<\omega}(C_{r,m}))$.

Consider the finite monogenic semigroup $C_{2,m} = \{a, a^2, \dots, a^{m+1} | a^{m+2} = a^2\}$. In this case, $xy \in C_m$ for any $x, y \in C_{2,m}$ and thus $\mathcal{A} \circ \mathcal{B} = N_{<\omega}\varphi(\mathcal{A}) \circ N_{<\omega}\varphi(\mathcal{B})$ for any $\mathcal{A}, \mathcal{B} \in N_{<\omega}(C_{2,m})$. It is easy to see that $e = a^m$ is the identity element of the maximal subgroup $C_m = C_{2,m} \setminus \{a\}$. Since $\varphi(a) = ea = a^m a = a^{m+1}$, then a^{m+1} is the unique element whose preimage under retraction $\varphi : C_{2,m} \to C_m$ is not a singleton. Therefore,

$$C(N_{<\omega}(C_{2,m})) = \{\langle\{a, a^{m+1}\}\rangle, \langle C_m\rangle, \langle C_{2,m}\rangle, \langle C_{2,m} \setminus \{a^{m+1}\}\rangle, \langle\{g\}\rangle : g \in C_{2,m}\}.$$

□

Problem 3.4. Given a monogenic semigroup $C_{r,m}$, $r > 2$, describe the center of the semigroup $N_{<\omega}(C_{r,m})$.

4. Right and Left Cancelable Elements

In this section, we shall detect right and left cancelable elements of extensions $N_{<\omega}(C_{r,m})$ of finite monogenic semigroups $C_{r,m}$.

Proposition 4.1. *The extension* $N_{<\omega}(C_{r,m})$ *has* (*left*, *right*) *cancelable elements if and only if the index r of a monogenic semigroup* $C_{r,m}$ *is equal to* 1.

Proof. Let $r > 1$ and a be the generator of a semigroup $C_{r,m}$. Consider the map $\varphi : C_{r,m} \to C_m$, $\varphi : x \to ex$, where e is the neutral element of the cyclic group C_m. As we showed in the proof of Theorem 3.3, $\varphi(a^{r-1})x = a^{r-1}x$ for any $x \in C_{r,m}$.

Let $\mathcal{M}$ be a centered upfamily on a semigroup $C_{r,m}$. Then we obtain $\langle\{a^{r-1}\}\rangle \circ \mathcal{M} = \left\langle \bigcup_{a \in \{a^{r-1}\}} a * M_a : \{M_a\}_{a \in \{a^{r-1}\}} \subset \mathcal{M} \right\rangle = \left\langle a^{r-1}M : M \in \mathcal{M} \right\rangle$ $= \left\langle \varphi(a^{r-1})M : M \in \mathcal{M} \right\rangle = \{\varphi(a^{r-1})\}\rangle \circ \mathcal{M}$ and $\mathcal{M} \circ \langle\{a^{r-1}\}\rangle =$ $\left\langle \bigcup_{a \in M} a * \{a^{r-1}\} : M \in \mathcal{M} \right\rangle = \left\langle Ma^{r-1} : M \in \mathcal{M} \right\rangle = \left\langle M\varphi(a^{r-1}) : M \in \mathcal{M} \right\rangle$ $= \mathcal{M} \circ \langle\{\varphi(a^{r-1})\}\rangle$. Since $a^{r-1} \neq \varphi(a^{r-1})$, the centered upfamily $\mathcal{M}$ is neither left nor right cancelable.

If $r = 1$, then a monogenic semigroup $C_{1,m} = C_m$ is a group. Let e be the neutral element of the group C_m. Then $\langle\{e\}\rangle \circ \mathcal{M} = \mathcal{M} = \mathcal{M} \circ \langle\{e\}\rangle$ for any $\mathcal{M} \in N_{<\omega}(C_m)$, and equalities $\chi \circ \langle\{e\}\rangle = \mathcal{Y} \circ \langle\{e\}\rangle$, $\langle\{e\}\rangle \circ \chi = \langle\{e\}\rangle \circ \mathcal{Y}$ imply that $\chi = \mathcal{Y}$. Consequently, the principal ultrafilter $\langle\{e\}\rangle$ is a cancelable element of the semigroup $N_{<\omega}(C_{1,m})$. □

If G is a group, then the formula

$$\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a \ : L \in \mathcal{L}, \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$$

implies that the product $\mathcal{L} \circ \mathcal{M}$ of any two centered upfamilies $\mathcal{L}$ and $\mathcal{M}$ is a principal ultrafilter if and only if both $\mathcal{L}$ and $\mathcal{M}$ are principal ultrafilters. Therefore, we deduce the following proposition:

Proposition 4.2. *For a group G, the set $N_{<\omega}(G) \setminus \{\langle\{g\}\rangle : g \in G\}$ is an ideal in $N_{<\omega}(G)$.*

Lemma 4.3. *A semigroup S is a left (right) cancellative semigroup if and only if all principal ultrafilters are left (right) cancelable elements in the extension $N_{<\omega}(S)$.*

Proof. If an element $a \in S$ is not left (right) cancelable in the semigroup S, then it is clear that the principal ultrafilter generated by the element a is not left (right) cancelable in $N_{<\omega}(S)$.

Let S be a left (right) cancellative semigroup, $a \in S$ and $\chi, \mathcal{Y} \in N_{<\omega}(S)$, $\chi \neq \mathcal{Y}$, then without loss of generality we can assume that $X \in \chi \setminus \mathcal{Y}$ for some $X \in \chi$. Therefore, $(S \setminus X) \cap Y \neq \emptyset$ for any $Y \in \mathcal{Y}$. Since each element of S is left (right) cancelable, then $(S \setminus aX) \cap aY \neq \emptyset$ $((S \setminus Xa) \cap Ya \neq \emptyset)$, and thus $\langle\{a\}\rangle \circ \chi \neq \langle\{a\}\rangle \circ \mathcal{Y}(\chi \circ \langle\{a\}\rangle \neq \mathcal{Y} \circ \langle\{a\}\rangle)$. Consequently, the left $l_{\langle\{a\}\rangle}$ (right $r_{\langle\{a\}\rangle}$) shift is injective and the principal ultrafilter $\langle\{a\}\rangle$ is left (right) cancelable. □

Proposition 4.4. *An element $\mathcal{M} \in N_{<\omega}(C_{1,m})$ is left (right) cancelable if and only if $\mathcal{M}$ is a principal ultrafilter.*

Proof. Since in any group, in particular in the cyclic group $C_{1,m}$, all elements are cancelable, then all principal ultrafilters are cancelable in the extension $N_{<\omega}(C_{1,m})$ according to Lemma 4.3.

Assume that some centered upfamily $\mathcal{M} \in N_{<\omega}(C_{1,m}) \setminus \{\langle\{g\}\rangle : g \in C_{1,m}\}$ is left cancelable. This means that the left shift $l_{\mathcal{M}} : N_{<\omega}(C_{1,m}) \to N_{<\omega}(C_{1,m})$, $l_{\mathcal{M}} : \mathcal{A} \mapsto \mathcal{M} \circ \mathcal{A}$, is injective. According to Proposition 4.2, the set $N_{<\omega}(C_{1,m}) \setminus \{\langle\{g\}\rangle : g \in C_{1,m}\}$ is an ideal in $N_{<\omega}(C_{1,m})$. Consequently, $l_{\mathcal{M}}(N_{<\omega}(C_{1,m})) = \mathcal{M} \circ N_{<\omega}(C_{1,m}) \subset N_{<\omega}(C_{1,m}) \setminus \{\langle\{g\}\rangle : g \in C_{1,m}\}$. Since $N_{<\omega}(C_{1,m})$ is finite, $l_{\mathcal{M}}$ cannot be injective.

For the right cancelable elements the proof is analogous. □

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