On superextensions of semigroups and their automorphism groups

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A family $\mathcal{M}$ of non-empty subsets of a set X is called *an upfamily* if for each set $A \in \mathcal{M}$ any subset $B \supset A$ of X belongs to $\mathcal{M}$. By $v(X)$ we denote the set of all upfamilies on a set X. Each family $\mathcal{B}$ of non-empty subsets of X generates the upfamily $\langle\mathcal{B}\rangle = \{A \subset X : \exists B \in \mathcal{B}\ (B \subset A)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called the *Stone-Čech compactification* of X, see [14]. An ultrafilter $\langle\{x\}\rangle$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle\{x\}\rangle$ generated by the singleton $\{x\}$, and hence we can consider $X \subset \beta(X) \subset v(X)$. It was shown in [8] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $* : v(S) \times v(S) \to v(S)$ by the formula

$$\mathcal{L} * \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle$$

for upfamilies $\mathcal{L}, \mathcal{M} \in v(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $v(S)$. The semigroup $v(S)$ contains as subsemigroups many other important extensions of S. In particular, it contains the semigroup $\lambda(S)$ of maximal linked upfamilies. An upfamily $\mathcal{L}$ of subsets of S is said to be *linked* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{L}$. A linked upfamily $\mathcal{M}$ of subsets of S is *maximal linked* if $\mathcal{M}$ coincides with each linked upfamily $\mathcal{L}$ on S that contains $\mathcal{M}$. It follows that $\beta(S)$ is a subsemigroup of $\lambda(S)$. The space $\lambda(S)$ is well-known in General and Categorial Topology as the *superextension* of S, see [16].

Given a semigroup S we shall discuss the algebraic structure of the automorphism group $\mathrm{Aut}\,(\lambda(S))$ of the superextension $\lambda(S)$ of S. We show that any automorphism of a semigroup S can be extended to an automorphism of its superextension $\lambda(S)$, and the automorphism group $\mathrm{Aut}\,(\lambda(S))$ of the superextension $\lambda(S)$ of a semigroup S contains a subgroup, isomorphic to the group $\mathrm{Aut}\,(S)$.

Proposition 1. *For any group G, each automorphism of $\lambda(G)$ is an extension of an automorphism of G.*

Theorem 1. *Two groups are isomorphic if and only if their superextensions are isomorphic.*

A semigroup S is called *monogenic* if it is generated by some element $a \in S$ in the sense that $S = \{a^n\}_{n\in\mathbb{N}}$. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$ of positive integer numbers. A finite monogenic semigroup $S = \langle a \rangle$ also has simple structure. There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{r+m-1}\}$ and $r + m - 1 = |S|$;
- $a^{r+m} = a^r$;
- $C_m := \{a^r, a^{r+1}, \dots, a^{r+m-1}\}$ is a cyclic and maximal subgroup of S with the neutral element $e = a^n \in C_m$ and generator a^{n+1}, where $n \in (m \cdot \mathbb{N}) \cap \{r, \dots, r + m - 1\}$.

By $\mathrm{M}_{r,m}$ we denote a finite monogenic semigroup of index r and period m.

Theorem 2. *Two finite monogenic semigroups are isomorphic if and only if their superextensions are isomorphic.*

Proposition 2. *If $r \geq 3$, then any automorphism ψ of the semigroup $\lambda(\mathrm{M}_{r,m})$ has $\psi(x) = x$ for all $x \in \mathrm{M}_{r,m}$.*

For the idempotent e of the maximal subgroup C_m of a semigroup $\mathrm{M}_{r,m}$ the shift $\rho : \mathrm{M}_{r,m} \to e\mathrm{M}_{r,m} = C_m$, $\rho : x \mapsto ex$, is a homomorphic retraction of $\mathrm{M}_{r,m}$ onto C_m. Therefore, $\bar{\rho} = \lambda\rho : \lambda(\mathrm{M}_{r,m}) \to \lambda(C_m) \subset \lambda(\mathrm{M}_{r,m})$ is a homomorphic retraction as well.

Theorem 3. *For $r = 2$ the homomorphic retraction $\bar{\rho} : \lambda(\mathrm{M}_{r,m}) \to \lambda(C_m)$ has the following properties:*

1. *$\mathcal{A} * \mathcal{B} = \bar\rho(\mathcal{A}) * \mathcal{B} = \mathcal{A} * \bar\rho(\mathcal{B}) = \bar\rho(\mathcal{A}) * \bar\rho(\mathcal{B})$ for any $\mathcal{A}, \mathcal{B} \in \lambda(\mathrm{M}_{r,m})$;*

2. *$\psi(x) = x$ for any $x \in C_m$ and any $\psi \in \mathrm{Aut}\,(\lambda(\mathrm{M}_{r,m}))$;*

3. *the restriction operator $R : \mathrm{Aut}\,(\lambda(\mathrm{M}_{r,m})) \to \mathrm{Aut}\,(\lambda(C_m))$ has kernel isomorphic to $\prod_{\mathcal{L}\in\lambda(C_m)} S_{\bar\rho^{-1}(\mathcal{L})\setminus\{\mathcal{L}\}}$ and the range*

$$R(\mathrm{Aut}\,(\mathrm{M}_{r,m})) = \{\varphi \in \mathrm{Aut}\,(\lambda(C_m)) : \forall \mathcal{L} \in \lambda(C_m) \ \ |\bar\rho^{-1}(\varphi(\mathcal{L}))| = = |\bar\rho^{-1}(\mathcal{L})|\}.$$

Consider the shift $\sigma : \mathrm{M}_{r,m} \to a\mathrm{M}_{r,m}$, $\sigma : x \mapsto ax$.

Theorem 4. *Assume that $r \geq 2$. The restriction operator $R : \mathrm{Aut}\,(\lambda(\mathrm{M}_{r,m})) \to \mathrm{Aut}\,(\lambda(\mathrm{M}_{r,m}^{\cdot 2}))$ has kernel isomorphic to*

$$\prod_{\mathcal{L}\in\lambda(\mathrm{M}_{r,m}^{\cdot 2})} S_{\bar\sigma^{-1}(\mathcal{L})\setminus\lambda(\mathrm{M}_{r,m}^{\cdot 2})}$$

and range $R(\mathrm{Aut}\,(\mathrm{M}_{r,m})) \subset H$ where

$$H = \big\{\varphi \in \mathrm{Aut}\,(\lambda(\mathrm{M}_{r,m}^{\cdot 2})) : \ \forall \mathcal{L} \in \lambda(\mathrm{M}_{r,m}^{\cdot 2}) \varphi(\bar\sigma^{-1}(\mathcal{L}) \cap \lambda(\mathrm{M}_{r,m}^{\cdot 2})) = = \bar\sigma^{-1}(\mathcal{L}) \cap \lambda(\mathrm{M}_{r,m}^{\cdot 2}) \text{ and } \forall C \in \Xi_{\lambda(\mathrm{M}_{r,m})} \ |\bar\sigma^{-1}(\varphi(\mathcal{L})) \cap C| = = |\bar\sigma^{-1}(\mathcal{L}) \cap C|\big\}.$$

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