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THE SPACES OF INCLUSION HYPERSPACES OVER NON-COMPACT SPACES

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We extend the construction of the space $G(X)$ of inclusion hyperspaces to non-compact spaces, prove the supercompactness of $G(X)$ for any T_1-space X, study the algebraic structure of $G(X)$, and define some important subspaces of $G(X)$.

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Мы распространяем конструкцию пространства $G(X)$ гиперпространств включения на некомпактные гиперпространства, доказываем суперкомпактность $G(X)$ для произвольного T_1-пространства X, изучаем алгебраическую структуру $G(X)$, и определяем некоторые важные подпространства $G(X)$.

The aim of this paper is to extend the construction of the space $G(X)$ of inclusion hyperspaces beyond the class of compact Hausdorff spaces, in which this construction is well-studied, see [12]. In a sense, this paper can be considered as a continuation of the work of E. Moiseev [11]. However our motivation came from the topological algebra where the obtained results will be applied, see [6], [1]. In this paper, given an arbitrary topological space X, we shall define the space $G(X)$ of inclusion hyperspaces on X and shall show that $G(X)$ is a supercompact T_1-space. Using the relation of $G(X)$ to the Wallman extension we show that the space $G(X)$ is Hausdorff if and only if X is normal (in which case $G(X)$ can be identified with $G(\beta X)$). By another method this equivalence was established also by E. Moiseev [11]. Also we introduce and study some important subspaces in $G(X)$: the space $\mathrm{Fil}(X)$ of filters, $N_k(X)$ of k-linked systems, $\lambda_k(X)$ of maximal k-linked systems.

Some preliminary information on hyperspaces. For a topological space X by $\exp(X)$ we denote the hyperspace of all non-empty closed subsets of X endowed with the Vietoris topology generated by the base consisting of the sets

$$\langle U_1,\dots,U_n\rangle=\{C\in\exp(X): C\subset\bigcup_{i=1}^n U_i \text{ and } \forall i\le n\ \ C\cap U_i\ne\varnothing\}$$

where $U_1,\dots,U_n$ run over open subsets of $X, n\in\mathbb{N}$.

Let us recall some known properties of the Vietoris topology. For a T_1-space X the hyperspace $\exp(X)$ is Hausdorff if and only if X is regular, see [5, §IV.3]. The normality of a T_1-space X is equivalent to the (complete) regularity of $\exp(X)$ while the compactness

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of X is equivalent to the normality (and compactness) of $\exp(X)$, see [4, 2.7.20, 3.12.26]. The Vietoris topology on the hyperspace $\exp(\mathbb{N})$ of the discrete space $\mathbb{N}$ is well-known in the Descriptive Set Theory as the Ellentuck topology. It is not normal, see [9, 19.D] or [4, 3.12.26]. Consequently the double hyperspace $\exp^2(\mathbb{N}) = \exp(\exp(\mathbb{N}))$ is not a regular space. In the sequel writing $C \underset{cl}{\subset} X$ we mean that C is a closed subset of a topological space X.

Inclusion hyperspaces. In this section we study inclusion hyperspaces on topological spaces and obtain a characterization theorem helping to recognize inclusion hyperspaces.

Definition 1. A subfamily $\mathcal{A} \subset \exp X$ is called *monotone* if $\mathcal{A} \ni A \subset B \underset{cl}{\subset} X$ implies $B \in \mathcal{A}$. Closed monotone subfamilies $\mathcal{A} \subset \exp X$ are referred to as *inclusion hyperspaces* on X.

Inclusion hyperspaces often appear as closures of monotone families in $\exp X$.

Proposition 1. *The closure* $\mathrm{cl}_{\exp X}(\mathcal{A})$ *of any monotone subset* $\mathcal{A} \subset \exp X$ *in* $\exp X$ *is an inclusion hyperspace on* X.

Proof. We have to prove that a closed subset $B \subset X$ belongs to $\mathrm{cl}_{\exp X}(\mathcal{A})$ provided it contains a subset $A \in \mathrm{cl}_{\exp X}(\mathcal{A})$. Let $\langle W_1, \dots, W_m \rangle$ be a basic neighborhood of the point $B \in \exp X$. Then $B \subset W_1 \cup \ldots \cup W_m$. Consider the sets W_i meeting the set A and write them as $W_{l_1}, \dots, W_{l_k}$. Then $A \in \langle W_{l_1}, \dots, W_{l_k} \rangle$. Since A is a cluster point of the set $\mathcal{A}$ in $\exp(X)$, there is $A' \in \mathcal{A} \cap \langle W_{l_1}, \dots, W_{l_k} \rangle$. The monotonicity of $\mathcal{A} \ni A'$ implies $A' \cup B \in \mathcal{A} \cap \langle W_1, \dots, W_n \rangle$ witnessing that B is a cluster point of $\mathcal{A}$ in $\exp X$. $\square$

This proposition implies that each non-empty family $\mathcal{B}$ of non-empty subsets of a topological space generates the inclusion hyperspace $\overline{\uparrow\mathcal{B}} = \underset{\exp(X)}{\mathrm{cl}} \{F \underset{cl}{\subset} X : \exists B \in \mathcal{B} \;\; B \subset F\}$.
A family $\mathcal{B}$ of subsets of X is called a *base* of an inclusion hyperspace $\mathcal{F}$ if $\mathcal{B} \subset \mathcal{F}$ and $\mathcal{F} = \overline{\uparrow\mathcal{B}}$.

The following characterization theorem allows us to define inclusion hyperspaces without appealing to the Vietoris topology.

Theorem 1 (Characterization Theorem). *A family $\mathcal{A}$ of non-empty closed subsets of a topological space X is an inclusion hyperspace if and only if $\mathcal{A}$ satisfies the condition: a closed subset $F \subset X$ belongs to $\mathcal{A}$ provided each neighborhood $U \subset X$ of F contains a set $A \in \mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is an inclusion hyperspace in X. We have to prove that a closed subset $F \subset X$ belongs to $\mathcal{A}$ provided each neighborhood U of $F \subset X$ contains a set $A \in \mathcal{A}$. For this it suffices to check that F is a cluster point of the set $\mathcal{A}$ in the hyperspace $\exp X$. For each basic neighborhood $\langle U_1, \dots, U_n \rangle$ of F in the space $\exp X$ the union $U_1 \cup \ldots \cup U_n$, being a neighborhood of F in X, contains some set $A \in \mathcal{A}$ according to our hypothesis. Then the closed set $F \cup A$ belongs to $\mathcal{A} \cap \langle U_1, \dots, U_n \rangle$ and hence F is a cluster points of $\mathcal{A}$.

Now assume that a family $\mathcal{A}$ of closed non-empty subsets of a topological space X satisfies the condition: a closed subset $F \subset X$ belongs to $\mathcal{A}$ provided each neighborhood U of F contains a set $A \in \mathcal{A}$. Let us show that $\mathcal{A}$ is an inclusion hyperspace.

At first, remark that the above condition implies that the family $\mathcal{A} \ni A$ is monotone. So it remains to check that $\mathcal{A}$ is closed in $\exp X$. Indeed, let $F \underset{cl}{\subset} X$ be a cluster point of $\mathcal{A}$ in $\exp X$. If $U \supset F$ is open, then $\langle U \rangle$ is a neighborhood of F in $\exp X$ containing some point $A \in \mathcal{A}$. Then $A \subset U$ and our condition guarantees that $F \in \mathcal{A}$. Thus the set $\mathcal{A}$ is an inclusion hyperspace, being a closed monotone family in $\exp(X)$. $\square$

Let us note that Proposition 1 can be deduced from the Characterization Theorem 1.

Remark 1. Since each subset of a discrete space is closed-and-open, in the discrete case the characterizing theorem 1 can be written in a simpler form: a family $\mathcal{A}$ of non-empty subsets of a discrete topological space X is an inclusion hyperspace if and only if $\mathcal{A}$ is monotone.

The space $G(X)$ and its topologization. By $G(X)$ we denote the set of all inclusion hyperspaces on a topological space X. Since the elements $\mathcal{A} \in G(X)$ are closed subsets of the hyperspace $\exp X$, the set $G(X)$ can be identified with a subspace of the double hyperspace $\exp^2(X)$ and endowed with the subspace topology. Such an approach works well for compact spaces X. Moreover, in this case the subspace topology on $G(X)$ is generated by the sub-base consisting of the sets $U^+ = \{\, \mathcal{A} \in G(X) \colon \exists A \in \mathcal{A} \;\; A \subset U \,\}$ and $U^- = \{\, \mathcal{A} \in G(X) \colon \forall A \in \mathcal{A} \;\; A \cap U \neq \varnothing \,\}$, where U runs over all open subsets of X. For non-compact X the Vietoris topology of the double hyperspace $\exp^2(X)$ has rather bad properties (for example, it is not regular even for the space $\exp^2(\mathbb{N})$).

So we topologize the space $G(X)$ directly (without appealing to the double hyperspace) endowing $G(X)$ with the topology generated by the subbase consisting of the sets U^+ and U^- where U runs over the topology of the space X. This topology on $G(X)$ has rather good properties.

Theorem 2. *$G(X)$ is a supercompact T_1-space for any topological space X.*

We recall that a space X is *supercompact* if any its cover by elements of a suitable sub-base has a binary subcover. Sub-bases with that property are called *binary*.

Proof. Firstly we prove that $G(X)$ is a T_1-space. Let $\mathcal{F},\mathcal{U} \in G(X)$, $\mathcal{F} \neq \mathcal{U}$. Without loss of generality we can assume that there is $F \in \mathcal{F} \backslash \mathcal{U}$. Since $F \notin \mathcal{U}$, the Characterization Theorem 1 yields a neighborhood $V \supset F$ containing no set $G \in \mathcal{U}$. Therefore $\mathcal{U} \notin V^+$ and $\mathcal{F} \in V^+$.

On the other hand, for any $G \in \mathcal{U}$ the set G does not lie in F, i.e., $G \cap (X \backslash F) \neq \varnothing$ and $\mathcal{U} \in (X\backslash F)^-$. From $F \cap (X\backslash F) = \varnothing$ it follows that $\mathcal{F} \notin (X\backslash F)^-$.

Now let us show that $G(X)$ is a supercompact space. Let $G(X)$ be the union of some sets from the canonical subbase $G(X) = \bigcup_{i\in I} U_i^- \cup \bigcup_{j\in J} V_j^+$.

If $X = U_i$ for some $i \in I$ or $X = V_j$ for some $j \in J$, then $U_i^- = G(X)$ or $V_j^+ = G(X)$ and we are done. So from now on, we assume that $X \neq U_i$ and $X \neq V_j$ for all $i \in I$, $j \in J$.

If $I = \varnothing$, then we get a cover $G(X) = \bigcup_{j\in J} V_j^+$. Taking into account that $V_j \neq X$ for every j we conclude that $\{X\} \in G(X)$, $\{X\} \notin \bigcup_{j\in J} V_j^+$ which is a contradiction. Thus $X = V_j$ for some j and $G(X) = V_j^+$.

So, we assume that $I \neq \varnothing$. Consider the family $\mathcal{F}' = \{F \in \exp(X) : F \supset X\backslash U_i$ for some $i \in I\}$ and its closure $\mathcal{F} = \mathrm{cl}_{\exp X} \mathcal{F}'$, which is an inclusion hyperspace according to Proposition 1.

It follows from $X \setminus U_i \in \mathcal{F}$ that $\mathcal{F} \notin U_i^-$ for all $i \in I$. Hence, $\mathcal{F} \in V_{j_0}^+$ for some $j_0 \in J$, which means that there is $F' \in \mathcal{F}$ with $F' \subset V_{j_0}$. Since

$$F' \in \underset{\exp X}{\mathrm{cl}} \{F \subset X : F \supset X\backslash U_i \text{ for some } i\} \cap \langle V_{j_0} \rangle,$$

there exists $F'' \in \langle V_{j_0}\rangle$ with $F'' \in \mathcal{F}'$. Therefore for some $i \in I$ we get $X\backslash U_i \subset F'' \subset V_{j_0}$ and thus $U_i \cup V_{j_0} = X$. Let us show that $U_i^- \cup V_{j_0}^+ = G(X)$. If $\mathcal{F} \notin V_{j_0}^+$, then for every $F \in \mathcal{F}$ the set F is not contained in V_j and thus $F \cap (X\backslash V_j) \neq \varnothing$. On the other hand, $X\backslash V_j \subset U_i$ implies $F \cap U_i \neq \varnothing$ and $\mathcal{F} \in U_i^-$.

□

We recall that a topological space X is a *T_4-space* if any disjoint closed subsets of X have disjoint open neighborhoods.

Proposition 2. *If X is a T_4-space, then $G(X)$ is a T_2-space.*

Proof. Let $\mathcal{F}_0, \mathcal{F}_1 \in G(X)$, $\mathcal{F}_0 \neq \mathcal{F}_1$. Without loss of generality we can assume that there is $F_0 \in \mathcal{F}_0\backslash\mathcal{F}_1$. According to the Characterization Theorem 1, there is a neighborhood W of the set F_0 containing no set from the inclusion hyperspace $\mathcal{F}_1$. Since X is a T_4-space, for the neighborhood W of the set F_0 there is a neighborhood $W_0 \supset F_0$ such that $\operatorname{cl} W_0 \subset W$ and thus $\mathcal{F}_0 \in W_0^+$. Let $W_1 = X\backslash \operatorname{cl} W_0$. Then $W_0 \cap W_1 = \varnothing$. We claim that $\mathcal{F}_1 \in W_1^-$. Indeed, in the opposite case there is $F \in \mathcal{F}_1$ such that $F \cap W_1 = \varnothing$. Thus $F \subset \operatorname{cl} W_0 \subset W$ and we obtain a contradiction with the choice of the neighborhood W.

Let us show that $W_0^+ \cap W_1^- = \varnothing$. If $\mathcal{F} \in W_0^+$, then there is $F \in \mathcal{F}$ such that $F \subset W_0$ and hence $F \cap (X\backslash W_0) = \varnothing$. Therefore $F \cap W_1 = \varnothing$ and $\mathcal{F} \notin W_1^-$. □

In Section "Relation to the Wallman extension" using the Wallman extension we partially reverse Proposition 2 and prove that for a T_1-space X the T_2-axiom of $G(X)$ is equivalent to the T_4-axiom of X.

Canonical embedding of X into $G(X)$. For a point x of a topological space X consider the monotone family $i_X(x) = \overline{\uparrow x} = \operatorname{cl}_{\exp X}\{F \subset_{cl} X : x \in F\}$, which is an inclusion hyperspace according to Proposition 1. In such a way we define a map $i_X : X \to G(X)$, $i_X : x \mapsto \overline{\uparrow x}$.

Proposition 3. *For a topological space X the following conditions are equivalent:*
1) X is a T_1-space; 2) $i_X : X \to G(X)$ is injective and continuous; 3) $i_X : X \to G(X)$ is a topological embedding.

Proof. (1) ⇒ (3) Assume that X is a T_1-space. First we prove that $i_X(x) = \{F \in \exp(X) : x \in F\}$ for every $x \in X$. This will follow as soon as we check that the family $\uparrow x = \{F \in \exp(X) : x \in F\}$ is closed in $\exp(X)$. Indeed, for every $F \in \exp(X) \setminus \uparrow x$ the set $\langle X \setminus \{x\}\rangle$ is an open neighborhood of F missing the set $\uparrow x$.

The formula $i_X(x) = \uparrow x$ implies that $i_X^{-1}(V^+) = V = i_X^{-1}(V^-)$ for every open set $V \subset X$. This equality can be used to prove that the map $i_X : X \to G(X)$ is a topological embedding.

The implication (3) ⇒ (2) is trivial while (2) ⇒ (1) follows from Theorem 2 and the fact that a topological space is T_1 if it admits an injective continuous map into a T_1-space. □

Let us present simple examples of spaces X for which i_X is not a topological embedding.

Example 1. Consider the topological space $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, X, \{b\}, \{a, b\}, \{b, c\}\}$. It satisfies the Axiom T_0 but not T_1. Then: $i_X(a) = \big\{\{a\}, \{a, c\}, \{a, b, c\}\big\}$, $i_X(b) = \big\{\{a, b, c\}\big\}$, $i_X(c) = \big\{\{c\}, \{a, c\}, \{a, b, c\}\big\}$, $i_X^{-1}(\{a, b\}^+) = \{a\}$, – is not open in X. Thus the map $i_X : X \to G(X)$ is injective and discontinuous.

Example 2. Consider the connected two-point space $X = \{a, b\}$ with the topology $\tau = \{\varnothing, \{a\}, \{a, b\}\}$. Then: $i_X(a) = \mathrm{cl}_{\exp X}\{\{a, b\}\} = \{\{b\}, \{a, b\}\}$, $i_X(b) = \mathrm{cl}_{\exp X}\{\{b\}, \{a, b\}\} = \{\{b\}, \{a, b\}\}$. This example yields a T_0-space X for which the map $i_X : X \to G(X)$ is continuous but not injective.

Example 3. The topological sum of the spaces considered in the preceding two examples yields a T_0-space X for which the map $i_X : X \to G(X)$ is discontinuous and non-injective.

Relation to the Wallman extension. In this section we show that for a T_1-space X the space $G(X)$ contains the Wallman extension ωX of X, which will allow us to use the known information on Wallman extensions to study the spaces $G(X)$. At first let us recall some definitions.

Definition 2. A family $\mathcal{F}$ of closed subsets of a topological space X is called a *filter of closed subsets* if the following conditions hold: **F1)** $\varnothing \notin \mathcal{F}$; **F2)** if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$; **F3)** if $F \in \mathcal{F}$ and $F \subset F' \underset{cl}{\subset} X$, then $F' \in \mathcal{F}$. A filter of closed subsets $\mathcal{F}$ on X is called *a closed filter* on X if $\mathcal{F}$ is a closed subset in $\exp X$.

Thus closed filters on X are inclusion hyperspaces closed under taking finite intersections.

Definition 3. *A filter of closed subsets of X which is maximal with respect to the inclusion relation is called an ultrafilter of closed subsets.*

Proposition 4. *Each ultrafilter of closed subsets on a T_1-space X is a closed filter and thus belongs to $G(X)$.*

Proof. Let $\mathcal{U}$ be an ultrafilter of closed subsets on X. To see that $\mathcal{U}$ is an inclusion hyperspace, we shall apply the Characterization Theorem 1. Let $F \subset X$ be a closed set whose any neighborhood contains a set from $\mathcal{U}$. We should prove that $F \in \mathcal{U}$. If it is not so, then the maximality of $\mathcal{U}$ would yield a subset $A \in \mathcal{U}$ missing the set F. Its complement $X \setminus A$ would be a neighborhood of F containing no subset from $\mathcal{U}$, which contradicts the choice of F. □

Definition 4. The set ωX of all ultrafilters of closed subsets with the topology generated by the base $\{U^+ \mid U \underset{op}{\subset} X\}$ is called *the Wallman extension* of the space X.

Proposition 4 allows us to think of the Wallman extension ωX as a subspace of the space $G(X)$.

Proposition 5. *If X is a T_1-space, then the Wallman extension ωX is a subspace of the space $G(X)$.*

Proof. The base of the topology on ωX consists of the sets: $U^+ = \{\, \mathcal{A} \in \omega X : \exists A \in \mathcal{A}, A \subset U \,\}$, where U runs over the family of open subsets of X. Let us prove that the topology of ωX coincides with the topology induced from $G(X)$. This will follows as soon as we check that $\omega X \cap U^- = \omega X \cap U^+$. Let us show that for every $\mathcal{U} \in \omega X \cap U^-$ there is $F \in \mathcal{U}$ with $F \subset U$, i.e., $\mathcal{U} \in \omega X \cap U^+$. Assume the opposite: $F \cap (X \backslash U) \neq \varnothing$ for every $F \in \mathcal{U}$. Then $X \backslash U \in \mathcal{U}$ and $(X \backslash U) \cap U = \varnothing$, a contradiction with $\mathcal{U} \in U^-$.

Let us check that $\omega X \cap U^+ \subset \omega X \cap U^-$. Fix any ultrafilter $\mathcal{U} \in \omega X \cap U^+$ and find a set $F' \in \mathcal{U}$ with $F' \subset U$. Then $\mathcal{U} \in U^-$ because each element of $\mathcal{U}$ meets the set F'. □

Therefore for a T_1-space X we obtain a chain of inclusions $X \subset \omega X \subset G(X)$.

Corollary 1. *For a T_1-space X the following conditions are equivalent: 1. X is normal; 2. ωX is Hausdorff; 3. $G(X)$ is Hausdorff.*

Proof. The implication (1) $\Rightarrow$ (3) is proved in Proposition 2, (3) $\Rightarrow$ (2) follows from the embedding $\omega X \subset G(X)$, and (2) $\Rightarrow$ (1) is proved in Theorem 3.6.22 of [4, c.273-274]. $\square$

Remark 2. The equivalence of the items (1) and (3) was first established by E. Moiseev [11] (by a different method).

Inclusion hyperspaces with finite support. In this section we study free inclusion hyperspaces and hyperspaces having finite support.

An inclusion hyperspace $\mathcal{A} \in G(X)$ is defined to have *finite support in* X if $\mathcal{A} = \overline{\uparrow\mathcal{F}}$ for some finite family $\mathcal{F}$ of finite subsets of X. By $G^\bullet(X)$ we denote the subspace of $G(X)$ that consists of the inclusion hyperspaces with finite support in X.

Proposition 6. *For a discrete space X the space $G^\bullet(X)$ is discrete.*

Proof. Let $\mathcal{F}$ be an inclusion hyperspace with the minimal finite base $\{F_1, F_2, \dots, F_n\}$ of finite subset. Then the open set $O(\mathcal{F}) = \bigcap_{i=1}^n \left(F_i^+ \cap \bigcap_{a\in F_i} ((X \setminus F_i) \cup \{a\})^-\right)$ coincides with the singleton $\{\mathcal{F}\}$. $\square$

Proposition 7. *For a T_1-space X the set $G^\bullet(X)$ is dense in $G(X)$.*

Proof. Fix an inclusion hyperspace $\mathcal{A} \in G(X)$ and a neighborhood $O(\mathcal{A})$ of $\mathcal{A}$ in $G(X)$. We can assume that $O(\mathcal{A})$ is of the basic form $O(\mathcal{A}) = U_1^+ \cap \dots \cap U_n^+ \cap V_1^- \cap \dots \cap V_m^-$ for some open sets $U_1, \dots, U_n$ and $V_1, \dots, V_m$ in X. For every $i \le n$ choose a closed set $A_i \in \mathcal{A}$ such that $A_i \subset U_i$. Since A_i meets each set V_j, $j \le m$, we may select a finite subset $F_i \subset A_i$ that meets all sets V_j, $j \le m$. Using the T_1-axiom of X it can be shown that the family $\mathcal{F} = \{F \underset{cl}{\subset} X : \exists i \le n \;\; F_i \subset F\}$ is an inclusion hyperspace with finite support and $\mathcal{F}$ belongs to the open neighborhood $O(\mathcal{A})$. $\square$

Next, we consider so called free inclusion hyperspaces. An inclusion hyperspace $\mathcal{F} \in G(X)$ on a non-compact space X is called *free* if for each compact subset $K \subset X$ and any element $F \in \mathcal{F}$ there is another element $E \in \mathcal{F}$ such that $E \subset F \setminus K$. By $G^\circ(X)$ we shall denote the subset of $G(X)$ consisting of free inclusion hyperspaces. It is clear that $G^\bullet(X) \cap G^\circ(X) = \varnothing$ for any T_1-space X.

In the simplest case of a countable discrete space $X = \mathbb{N}$ free inclusion hyperspaces (called semifilters) on $X = \mathbb{N}$ have been introduced and intensively studied in [2].

Proposition 8. *For a locally compact normal space X the set $G^\circ(X)$ of free inclusion hyperspaces is closed and nowhere dense in $G(X)$.*

Proof. Fix any inclusion hyperspace $\mathcal{F} \in G(X) \setminus G^\circ(X)$. Then there is a closed set $F \in \mathcal{F}$ and a compact set $K \subset X$ such that $F' \cap K \neq \varnothing$ for each set $F' \in \mathcal{F}$ with $F' \subset F$. Since X is locally compact, we can find an open neighborhood $V \subset X$ of K whose closure $\overline{V}$ in X is compact. It follows that $F \setminus V \notin \mathcal{F}$. Using Theorem 1 we can find a neighborhood $O(F \setminus V)$ of $F \setminus V$ that contains no subset $F' \in \mathcal{F}$. By the normality of X there are open sets $U, W \subset X$ such that $F \subset U \subset \overline{U} \subset W \subset \overline{W} \subset O(F \setminus V) \cup V$. Consider the open set

$\mathcal{U} = U^+ \cap (V \cup (X \setminus \overline{W}))^-$ in $G(X)$. We claim that $\mathcal{F} \in \mathcal{U}$. Indeed, $\mathcal{F} \in U^+$ because $F \subset U$ and $F \in \mathcal{F}$. Next, assuming that some $F' \in \mathcal{F}$ does not intersect the set $V \cup (X \setminus \overline{W})$ we conclude that $F' \subset \overline{W} \setminus V \subset O(F \setminus V)$ which is not possible. Thus $\mathcal{U}$ is an open neighborhood of the inclusion hyperspace $\mathcal{F}$ in $G(X)$.

We claim that $\mathcal{U} \cap G^\circ(X) = \varnothing$. Assuming the converse we would find a free inclusion hyperspace $\mathcal{E} \in \mathcal{U}$. Since $\mathcal{E} \in U^+$ there is a set $E \in \mathcal{E}$ with $E \subset U$. Since $\overline{V}$ is compact and $\mathcal{E}$ is free, there is a set $E' \in \mathcal{E}$ such that $E' \subset E \setminus \overline{V} \subset U \setminus V \subset \overline{W} \setminus V$. Then the set E' has empty intersection with $V \cup (X \setminus \overline{W})$, which contradicts the inclusion $\mathcal{E} \in \mathcal{U}$.

The subset $G^\circ(X)$ is nowhere dense in $G(X)$, being closed and disjoint with the dense subset $G^\bullet(X)$. □

The inner algebraic structure of $G(X)$. In this section we study the algebraic structure of the spaces $G(X)$. For every topological space X the space $G(X)$ possesses two binary operations

$$\cup : G(X) \times G(X) \to G(X), \quad \cup : (\mathcal{F},\mathcal{U}) \mapsto \mathcal{F} \cup \mathcal{U},$$

$$\cap : G(X) \times G(X) \to G(X), \quad \cap : (\mathcal{F},\mathcal{U}) \mapsto \mathcal{F} \cap \mathcal{U},$$

and one unary operation $\perp\colon G(X) \to G(X)$, $\perp\colon \mathcal{F} \mapsto \mathcal{F}^\perp = \{E \underset{cl}{\subset} X : \forall F \in \mathcal{F} \;\; E \cap F \neq \varnothing\}$ called the *transversality operation*.

Let us check that $\mathcal{F}^\perp$ indeed is an inclusion hyperspace. It is clear that it is a monotone family in $\exp(X)$. To show that $\mathcal{F}^\perp$ is closed in $\exp(X)$ take any closed subset $E \in \exp(X) \setminus \mathcal{F}^\perp$ and find $F \in \mathcal{F}$ with $E \cap F = \varnothing$. Then $\langle X \setminus F\rangle$ is an open neighborhood of E in $\exp(X)$ missing the set $\mathcal{F}^\perp$. The same argument shows that for every family $\mathcal{F} \subset \exp(X)$ the transversal family $\mathcal{F}^\perp = \{E \underset{cl}{\subset} X : \forall F \in \mathcal{F} \;\; E \cap F \neq \varnothing\}$ is an inclusion hyperspace. Moreover, the double transversal $\mathcal{F}^{\perp\perp}$ coincides with the inclusion hyperspace $\overline{\uparrow\mathcal{F}} = \mathrm{cl}_{\exp(X)}\{E \underset{cl}{\subset} X : \exists F \in \mathcal{F} \;\; F \subset E\}$ generated by $\mathcal{F}$.

Proposition 9. *For any non-empty family $\mathcal{F} \subset \exp(X)$ we get $\mathcal{F}^{\perp\perp} = \overline{\uparrow\mathcal{F}}$.*

Proof. The inclusion $\overline{\uparrow\mathcal{F}} \subset \mathcal{F}^{\perp\perp}$ follows from the definitions of $\mathcal{F}^\perp$, $\mathcal{F}^{\perp\perp}$, and the closedness of $\mathcal{F}^{\perp\perp}$ in $\exp(X)$.

Assuming that $\overline{\uparrow\mathcal{F}} \neq \mathcal{F}^{\perp\perp}$, find a set $F \in (\mathcal{F}^\perp)^\perp \setminus \overline{\uparrow\mathcal{F}}$. The non-inclusion $F \notin \overline{\uparrow\mathcal{F}}$ combined with Theorem 1 implies the existence of a neighborhood U of the set F containing no set $E \in \mathcal{F}$. It follows that $(X \setminus U) \in \mathcal{F}^\perp$. Since $F \in (\mathcal{F}^\perp)^\perp$, we obtain $F \cap (X \setminus U) \neq \varnothing$, which is a contradiction. □

Some basic properties of the transversality map are presented in

Proposition 10. *Let $\mathcal{F},\mathcal{U} \in G(X)$ be inclusion hyperspaces and $U \subset X$ be an open subset. Then: 1. $(\mathcal{F}^\perp)^\perp = \mathcal{F}$; 2. $\mathcal{F} \in U^+$ if and only if $\mathcal{F}^\perp \in U^-$; 3. $(\mathcal{F} \cup \mathcal{U})^\perp = \mathcal{F}^\perp \cap \mathcal{U}^\perp$; 4. $(\mathcal{F} \cap \mathcal{U})^\perp = \mathcal{F}^\perp \cup \mathcal{U}^\perp$.*

Proof. 1. The equality $(\mathcal{F}^\perp)^\perp = \mathcal{F}$ follows from the equalities $\mathcal{F}^{\perp\perp} = \overline{\uparrow\mathcal{F}} = \mathcal{F}$.

2. If $\mathcal{F} \in U^+$ then $F \subset U$ for some $F \in \mathcal{F}$ and then each set $E \in \mathcal{F}^\perp$ meets both F and U, which means that $\mathcal{F}^\perp \in U^-$. If $\mathcal{F} \notin U^+$, then each element $F \in \mathcal{F}$ meets the set $X \setminus U$ which yields $X \setminus U \in \mathcal{F}^\perp$ and hence $\mathcal{F}^\perp \notin U^-$.

3. The equality $(\mathcal{F} \cup \mathcal{U})^\perp = \mathcal{F}^\perp \cap \mathcal{U}^\perp$ easily follows from the definition of the transversality operation.

4. The preceding items imply $\mathcal{F}\cap\mathcal{U} = \mathcal{F}^{\perp\perp}\cap\mathcal{U}^{\perp\perp} = (\mathcal{F}^{\perp}\cup\mathcal{U}^{\perp})^{\perp}$. Applying to this equality the transversality operation we get $(\mathcal{F}\cap\mathcal{U})^{\perp} = (\mathcal{F}^{\perp}\cup\mathcal{U}^{\perp})^{\perp\perp} = \mathcal{F}^{\perp}\cup\mathcal{U}^{\perp}$. □

We apply this proposition to prove

Theorem 3. *The operations $\cup, \cap$ and $\perp$ on $G(X)$ are continuous.*

Proof. 1. The continuity of the transversality map follows from Proposition 10(2).

2. To see that the operation $\cup : G(X)\times G(X)\to G(X)$ is continuous, take any open set $U\subset X$ and note that $\{(\mathcal{F},\mathcal{U})\in G(X)\times G(X) : \mathcal{F}\cup\mathcal{U}\in U^{+}\} = U^{+}\times G(X)\cup G(X)\times U^{+}$ and $\{(\mathcal{F},\mathcal{U})\in G(X)\times G(X) : \mathcal{F}\cup\mathcal{U}\in U^{-}\} = U^{-}\times U^{-}$.

3. The continuity of the operation $\cap$ can be derived from the continuity of $\cup$ and $\perp$ with help of Proposition 10(4). □

The preceding propositions imply that for every topological space X the space $G(X)$ carries the structure of a symmetric lattice.

Definition 5. A *symmetric lattice* is a complete distributive lattice $(L,\vee,\wedge)$ endowed with an additional unary operation $\perp\colon L\to L$, $\perp\colon x\mapsto x^{\perp}$, that is an involutive anti-isomorphism in the sense that

- $x^{\perp\perp} = x$ for all $x\in L$;
- $(x\vee y)^{\perp} = x^{\perp}\wedge y^{\perp}$;
- $(x\wedge y)^{\perp} = x^{\perp}\vee y^{\perp}$.

The notion of a symmetric lattice was introduced in [2, §4.2].

For a discrete space X the set $G(X)$ of all inclusion hyperspaces on X is a subset of the double power-set $\mathcal{P}(\mathcal{P}(X))$ (which is a complete distributive lattice) and is closed under the operations of union and intersection (of arbitrary families of inclusion hyperspaces).

Since each inclusion hyperspace is a union of filters and each filter is an intersection of ultrafilters, we obtain the following proposition showing that the lattice $G(X)$ is a rather natural object.

Proposition 11. *For a discrete space X the lattice $G(X)$ coincides with the smallest complete sublattice of $\mathcal{P}(\mathcal{P}(X))$ containing all ultrafilters.*

Characterizing inclusion hyperspaces with finite supports. In this section we shall give a dual characterization of inclusion hyperspaces with finite support. This characterization will be essentially used in [6] for describing the topological center of the semigroup $G(X)$ over a discrete semigroup X.

Theorem 4. *An inclusion hyperspace $\mathcal{F}$ on a T_1-space X has finite support if and only if both $\mathcal{F}$ and $\mathcal{F}^{\perp}$ have bases consisting of finite sets.*

Proof. The "only if" part easily follows from the definition of an inclusion hyperspace with finite support. To prove the "if" part, assume that $\mathcal{F}$ and $\mathcal{F}^{\perp}$ have bases of finite subsets. We shall say that a subset $F\subset X$ is *$\mathcal{F}$-minimal* if $F\in\mathcal{F}$ but no proper subset $E\subset F$ belongs to $\mathcal{F}$. Since $\mathcal{F}$ has a base consisting of finite sets, the family $\mathcal{M}$ of $\mathcal{F}$-minimal sets contains only finite sets and is a base for $\mathcal{F}$. To show that $\mathcal{F}$ has finite support it suffices to check that the family $\mathcal{M}$ is finite.

Assume conversely that $\mathcal{M}$ is infinite. If $\mathcal{M}$ is uncountable, then by the Δ-Lemma [8], there is an uncountable subfamily $\mathcal{A} \subset \mathcal{B}$ and a set A_0 such that $A \cap A' = A_0$ for any distinct elements $A, A' \in \mathcal{A}$. The $\mathcal{F}$-minimality of each set $A \in \mathcal{A}$ implies that $A_0 \notin \mathcal{F}$ and thus $X \setminus A_0 \in \mathcal{F}^\perp$. Since $\mathcal{F}^\perp$ has a base consisting of finite sets, there is a finite subset $E \in \mathcal{F}^\perp$ with $E \subset X \setminus A_0$. Taking into account that the family $\{A \setminus A_0 : A \in \mathcal{A}\}$ is disjoint and each set $A \in \mathcal{A}$ meets the finite set $E = E \setminus A_0$, we arrive to an absurd conclusion that the family $\mathcal{A}$ is finite.

So it remains to consider the case of countable family $\mathcal{M}$. In this case the set $M = \bigcup \mathcal{M}$ is countable and thus admits a well-ordering $\le$ such that for every $x \in \bigcup \mathcal{M}$ the initial interval ${\downarrow}x = \{y \in M : y \le x\}$ is finite. Consider the family of finite sets ${\downarrow}\mathcal{M} = \{B \cap {\downarrow}x : B \in \mathcal{M},\ x \in M\}$. Endowed with the inclusion partial order, this family forms a tree T with finite branches whose maximal elements are sets from the family $\mathcal{M}$.

Let us show that each element $A \in {\downarrow}\mathcal{M}$ of this tree has at most finitely many immediate successors in the tree T. If A is a maximal element of the tree T, then A has no successors. So we assume that A is not a maximal element and hence A is a proper subset of some $B \in \mathcal{M}$. The $\mathcal{F}$-minimality of B implies that $A \notin \mathcal{F}$ and hence $X \setminus A \in \mathcal{F}^\perp$. Since $\mathcal{F}^\perp$ has a base of finite sets, there is a finite set $E \in \mathcal{F}^\perp$ with $E \subset X \setminus A$ and maximal element $e = \max E$. Now take any immediate successor $S \in {\downarrow}\mathcal{M}$ of A in T. Find a set $B \in \mathcal{M}$ and a point $b \in B$ such that $S = B \cap {\downarrow}b$. Being an immediate successor of A, the set S is equal to $A \cup \{b\}$. Taking into account that $(B \setminus A) \cap E = B \cap (E \setminus A) = B \cap E \neq \varnothing$ we see that $b = \min B \setminus A \le \max E = e$ and hence $S = A \cup \{b\} \subset {\downarrow}e$. Since the initial interval ${\downarrow}e$ is finite as well as its power-set, the family of all immediate successors of A in the tree T is finite. Therefore T is a tree with finite branches and finite immediate successors of each its element. By the König Lemma [8], the tree $T = {\downarrow}\mathcal{M}$ is finite and so is the set $\mathcal{M}$ of maximal elements of T. $\square$

Some important subspaces of $G(X)$. The space $G(X)$ of inclusion hyperspaces contains many interesting subspaces. In Section we have already met one of such subspaces, the Wallman extension ωX of X. In this section we shall consider some other subspaces of $G(X)$.

Let X be a topological space and $k \ge 2$ be a natural number. An inclusion hyperspace $\mathcal{A} \in G(X)$ is defined to be

- *k-linked* if $\cap\mathcal{F} \neq \varnothing$ for any subfamily $\mathcal{F} \subset \mathcal{A}$ with $|\mathcal{F}| \le k$;
- *centered* if $\cap\mathcal{F} \neq \varnothing$ for any finite subfamily $\mathcal{F} \subset \mathcal{A}$;
- *a filter* if $A_1 \cap A_2 \in \mathcal{A}$ for all sets $A_1, A_2 \in \mathcal{A}$.

By $N_k(X)$, $N_{<\omega}(X)$, and $\mathrm{Fil}(X)$ we denote the subsets of $G(X)$ consisting of k-linked, centered, and filter inclusion hyperspaces, respectively. It is clear that

$$\mathrm{Fil}(X) \subset N_{<\omega}(X) \subset N_k(X) \subset N_2(X) \subset G(X).$$

Now we show that for a normal space X all these sets are closed in $G(X)$.

Proposition 12. *For a normal space X the set of filters $\mathrm{Fil}(X)$ is closed in $G(X)$.*

Proof. Take any inclusion hyperspace $\mathcal{A} \in G(X) \setminus \mathrm{Fil}(X)$. Since $\mathcal{A} \notin \mathrm{Fil}(X)$, there are two sets $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 \notin \mathcal{A}$. Since $\mathcal{A}$ is monotone and closed in $\exp X$, there is a neighborhood $U \subset X$ of $A_1 \cap A_2$ such that $\mathcal{A} \notin U^+$. By the normality of X find a neighborhood W of $A_1 \cap A_2$ such that $\overline{W} \subset U$ and observe that $\mathcal{A} \in G(X) \setminus U^+ \subset (X \setminus \overline{W})^-$. Using the normality of X once more, find two open sets $U_1 \ni A_1$ and $U_2 \supset A_2$ such that $U_1 \cap U_2 \subset W$.

We claim that $U_1^+ \cap U_2^+ \cap (X \setminus \overline{W})^-$ is a neighborhood of $\mathcal{A}$ in $G(X)$ that misses the set $\mathrm{Fil}(X)$. Indeed, assuming that $U_1^+ \cap U_2^+ \cap (X \setminus \overline{W})^-$ contains some filter $\mathcal{F}$, find two sets $F_1, F_2 \in \mathrm{Fil}(X)$ with $F_1 \subset U_1$, $F_2 \subset U_2$. Then $\mathcal{F} \ni F_1 \cap F_2 \subset U_1 \cap U_2 \subset W$. On the other hand, $\mathcal{F} \subset (X \setminus \overline{W})^-$ implies that $F_1 \cap F_2 \not\subset \overline{W}$ which is a contradiction. □

Proposition 13. *For a normal space X the spaces $N_k(X)$ are closed in $G(X)$ for all $k \geq 2$.*

Proof. We first prove by induction that for the sets $F_1, \dots, F_k$ with empty intersection $F_1 \cap \dots \cap F_k$ there exist open sets $V_1 \supset F_1, \dots, V_k \supset F_k$ such that $V_1 \cap \dots \cap V_k$ is empty too. For $n = 2$ this follows from the definition of a normal space. Assuming that for $n = k$ the proposition has been proved we shall show that it is true also for $n = k + 1$. Let $F_1 \cap \dots \cap F_{k+1} = \varnothing$. For the sets $F_1 \cap \dots \cap F_k$ and F_{k+1} choose neighborhoods U_0 and U_{k+1} such that $U_0 \supset F_1 \cap \dots \cap F_k$, $U_{k+1} \supset F_{k+1}$ and $U_0 \cap U_{k+1} = \varnothing$. Denote by G_i the set $F_i \setminus U_0$, $i = 1, \dots k$. We have $G_1 \cap \dots \cap G_k = \varnothing$ and so can use the inductive assumption to select neighborhoods $V_i \supset G_i$ such that $V_1 \cap \dots \cap V_k = \varnothing$. Put $U_i = V_i \cup U_0$. We obtain that $U_i \supset F_i$ and $U_1 \cap \dots \cap U_k \cap U_{k+1} = ((V_1 \cup U_0) \cap \dots \cap (V_k \cup U_0)) \cap U_{k+1} = ((V_1 \cap \dots \cap V_k) \cup U_0) \cap U_{k+1} = U_0 \cap U_{k+1} = \varnothing$.

Now prove closedness $N_k(X)$ in $G(X)$. Let $\mathcal{F} \in \mathrm{cl}_{GX} N_k(X)$. Assume that $\mathcal{F}$ is not k-linked. Consequently there exist the sets $F_1, \dots, F_k \in \mathcal{F}$ with empty intersection. Choose neighborhoods $U_1 \supset F_1, \dots, U_k \supset F_k$ such that $U_1 \cap \dots \cap U_k = \varnothing$, then $\mathcal{F} \in U_1^+ \cap \dots \cap U_k^+$. Thus there exists $\mathcal{A} \in N_k X$ such that $\mathcal{A} \in U_1^+ \cap \dots \cap U_k^+$. Choose sets $A_1, \dots, A_k \in \mathcal{A}$ with $A_i \subset U_i$ for all $i \leq k$. Then $A_1 \cap \dots \cap A_k \subset U_1 \cap \dots \cap U_k = \varnothing$ contradicting the fact that $\mathcal{A}$ is k-linked. □

Since $N_{<\omega}(X) = \bigcap_{k \geq 2} N_k(X)$ the preceding proposition implies

Corollary 2. *For a normal space X the set $N_{<\omega}(X)$ of centered inclusion hyperspaces is closed in $G(X)$.*

An inclusion hyperspace $\mathcal{A} \in G(X)$ is called

- *maximal k-linked* if $\mathcal{A}$ is k-linked and $\mathcal{A} = \mathcal{B}$ for every k-linked inclusion hyperspace $\mathcal{B} \in N_k(X)$ with $\mathcal{A} \subset \mathcal{B}$;
- *an ultrafilter* if $\mathcal{A}$ is a filter and $\mathcal{A} = \mathcal{B}$ for every filter $\mathcal{B} \in \mathrm{Fil}(X)$ with $\mathcal{A} \subset \mathcal{B}$.

By ωX and $\lambda_k(X)$ we denote the subsets of $G(X)$ consisting of ultrafilters and maximal k-linked inclusion hyperspaces. According to Proposition 5, for a T_1-space the space ωX coincides with the Wallman extension of X and for a normal X coincides with the Stone-Čech compactification βX of X. The space $\lambda_2(X)$ usually is denoted by λX and is called the *superextension* of X, see [10].

Proposition 14. *For a normal space X the sets ωX and λX are closed in $G(X)$.*

Proof. By Corollary 1 the space $G(X)$ is Hausdorff. Observe that an inclusion hyperspace $\mathcal{A} \in G(X)$ is maximal 2-linked if and only if $\mathcal{A} = \mathcal{A}^{\perp}$. Consequently λX is closed as the set of fixed points of the transversality map $\perp\colon G(X) \to G(X)$ on the Hausdorff space $G(X)$. The set ωX is closed, being the intersection $\omega(X) = \mathrm{Fil}(X) \cap \lambda X$ of two closed sets. □

Remark 3. In light of the preceding proposition it is interesting to remark that for $k \geq 3$ the subspaces $\lambda_k(\mathbb{N})$ are not closed in $G(\mathbb{N})$, see [7].

It is well known (and easy to show) that each ultrafilter is maximal 2-linked. For maximal k-linked inclusion hyperspaces with $k > 2$ the situation is different:

Proposition 15. *If a maximal k-linked inclusion hyperspace $\mathcal{F}$ is $(k+1)$-linked, then it is an ultrafilter. Consequently, $\lambda_k(X) \cap N_{k+1}(X) \subset \omega X$.*

Proof. Let $A, B \in \mathcal{F}$. It is sufficient to prove that the intersection $A \cap B$ is in $\mathcal{F}$. The $(k+1)$-linkedness of $\mathcal{F}$ implies that for any $F_1, \dots, F_{k-1} \in \mathcal{F}$ the set $(A \cap B) \cap F_1 \cap \dots \cap F_{k-1} = A \cap B \cap F_1 \cap \dots \cap F_{k-1}$ is nonempty. Since $\mathcal{F}$ is maximal k-linked system, $A \cap B \in \mathcal{F}$. $\square$

Thus we obtain the following diagram describing the inclusion relations between the subspaces N_kX, $N_{<\omega}X$, $\mathrm{Fil}(X)$, λ_kX and ωX of $G(X)$ (an arrow $A \to B$ means that A is a subset of B):

$$\begin{tikzcd}
\mathrm{Fil}(X) \arrow[r] & N_{<\omega}X \arrow[r] & N_{k+1}X \arrow[r] & N_kX \arrow[r] & N_2X \arrow[r] & G(X) \\
& & N_{k+1}X \cap \lambda_kX \arrow[u] \arrow[r] \arrow[dll] & \lambda_kX \arrow[u] & & \\
\omega X \arrow[uu] \arrow[rrrr] & & & & \lambda X \arrow[uu] &
\end{tikzcd}$$

For any topological space X we can consider the intersections $\mathrm{Fil}^\bullet(X) = \mathrm{Fil}(X) \cap G^\bullet(X)$, $N^\bullet_{<\omega}(X) = N_{<\omega}(X) \cap G^\bullet(X)$, $N^\bullet_k(X) = N_k(X) \cap G^\bullet(X)$, $\lambda^\bullet(X) = \lambda X \cap G^\bullet(X)$ of the closed subsets of $G(X)$ with the dense subspace $G^\bullet(X)$ of inclusion hyperspaces with finite support.

The following proposition can be proved by a modification of the argument of the proof of Proposition 7.

Proposition 16. *For any T_1-space and every $k \geq 2$ the subspace $N^\bullet_k(X)$ (resp. $N^\bullet_{<\omega}(X)$, $\mathrm{Fil}^\bullet(X)$) is dense in $N_k(X)$ (resp. $N_{<\omega}(X)$, $\mathrm{Fil}(X)$).*

For a non-compact space X we can also consider the intersections

$$\begin{aligned}
\mathrm{Fil}^\circ(X) =& \mathrm{Fil}(X) \cap G^\circ(X), \quad N^\circ_{<\omega}(X) = N_{<\omega}(X) \cap G^\circ(X),\\
N^\circ_k(X) =& N_k(X) \cap G^\circ(X), \quad \lambda^\circ(X) = \lambda X \cap G^\circ(X), \text{ and}\\
\omega^\circ(X) =& \omega X \cap G^\circ(X) = \omega X \setminus X
\end{aligned}$$

of the closed subsets of $G(X)$ with the subspace $G^\circ(X)$ of free inclusion hyperspaces. The elements of those sets will be called free filters, free centered inclusion hyperspaces, free k-linked inclusion hyperspaces, etc.

Combining Propositions 12–14 with Proposition 8 we get

Corollary 3. *For a locally compact normal space X the subsets $\mathrm{Fil}^\circ(X)$, $N^\circ_{<\omega}(X)$, $N^\circ_k(X)$, $\lambda^\circ(X)$, $\omega^\circ(X)$ are closed in $G(X)$.*

In fact, the spaces λX and $\lambda^\circ(X)$ are even supercompact, see [10]. We include a short proof of this important property of λX and $\lambda^\circ(X)$ for completeness.

Proposition 17. *For any topological space X the superextension λX is supercompact.*

Proof. Taking into account item 2 of Proposition 10 and the equality $\mathcal{F} = \mathcal{F}^{\perp}$ holding for any $\mathcal{F} \in \lambda X$ we conclude that $\lambda X \cap U^+ = \lambda X \cap U^-$ for any open subset $U \subset X$. So it suffices to show that each cover $\{U_\alpha^- : \alpha \in A\}$ of λX has a two-element subcover. We claim that $X = U_\alpha \cup U_\beta$ for some indices $\alpha, \beta \in A$. Assuming that it is not so, we get that the family $\{X \setminus U_\alpha : \alpha \in A\}$ is 2-linked and hence can be enlarged to a maximal 2-linked inclusion hyperspace $\mathcal{L}$. It is clear that $\mathcal{L} \notin \bigcup_{\alpha\in A} U_\alpha^-$ which contradicts the fact that $\{U_\alpha^- : \alpha \in A\}$ is a cover of λX. Hence $X = U_\alpha \cup U_\beta$ for some indices $\alpha, \beta \in A$. This equality implies that $\lambda(X) \in U_\alpha^- \cup U_\beta^-$. □

Proposition 18. *For any locally compact space X the space $\lambda^\circ(X)$ is supercompact.*

Proof. We have to prove that each cover $\{U_\alpha^- : \alpha \in \Omega\}$ of $\lambda^\circ(X)$ by sub-basic sets has a two-element subcover. Let $\mathcal{K}(X)$ be the family of open subsets having compact closures in X.

We claim that $X = U_\alpha \cup U_\beta \cup K$ for some indices $\alpha, \beta \in \Omega$ and some subset $K \in \mathcal{K}(X)$. Assuming that it is not so, we get that the family $\{X \setminus (U_\alpha \cup K) : \alpha \in \Omega,\ K \in \mathcal{K}(X)\}$ is free and 2-linked. Applying the Zorn Lemma we can enlarge this family to a maximal 2-linked free inclusion hyperspace $\mathcal{L}$. It can be shown that $\mathcal{L}$ belongs to $\lambda^\circ(X)$ and hence $\mathcal{L} \in U_\alpha^-$ for some $\alpha \in \Omega$. On the other hand, this is not possible because $X \setminus U_\alpha \in \mathcal{L}$.

This contradiction shows that $X = U_\alpha \cup U_\beta \cup K$ for some $\alpha, \beta \in \Omega$ and $K \in \mathcal{K}(X)$. We claim that $\lambda^\circ(X) \subset U_\alpha^- \cup U_\beta^-$. Assuming the converse, we would find a free maximal linked inclusion hyperspace $\mathcal{F} \notin U_\alpha^- \cup U_\beta^-$, which contains two sets $A, B \in \mathcal{F}$ such that $A \cap U_\alpha = \varnothing$ and $B \cap U_\beta = \varnothing$. Then $A \cap B \subset X \setminus (U_\alpha \cup U_\beta) \subset K$.

Since the inclusion hyperspace $\mathcal{F}$ is free and K has compact closure in X, there are sets $A', B' \in \mathcal{F}$ such that $A' \subset A \setminus K$ and $B' \subset B \setminus K$. These sets have empty intersection $A' \cap B' \subset (A \cap B) \setminus K = \varnothing$ which contradicts to the choice of $\mathcal{F}$ as a linked inclusion hyperspace. □

Maps between spaces of inclusion hyperspaces. It is known that the construction of the space of inclusion hyperspaces is functorial in the category of compact Hausdorff spaces. In this section we shall extend this construction onto the category of normal topological spaces and on this base will show that for a normal space X the space $G(X)$ is canonically homeomorphic to $G(\beta X)$ where $\beta X = \omega X$ is the Stone-Čech compactification of X.

For any function $f : X \to Y$ between topological spaces consider the function $Gf : G(X) \to G(Y)$ assigning to an inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

$$Gf(\mathcal{F}) = f(\mathcal{F})^{\perp\perp} = \operatorname*{cl}_{\exp(Y)}(f(\mathcal{F}))$$

where $f(\mathcal{F}) = \{E \underset{cl}{\subset} Y : \exists F \in \mathcal{F} \text{ with } F \subset f^{-1}(E)\} \subset \exp(Y)$ (the equality $f(\mathcal{F})^{\perp\perp} = \overline{f(\mathcal{F})}$ in the definition of $Gf(\mathcal{F})$ follows from Proposition 9).

The so-defined function $Gf : G(X) \to G(Y)$ is a lattice homomorphism.

Proposition 19. *Let $f : X \to Y$ be a function between topological spaces and $\mathcal{A}, \mathcal{B} \in G(X)$ be two inclusion hyperspaces. Then*

1. $Gf(\mathcal{A} \cup \mathcal{B}) = Gf(\mathcal{A}) \cup Gf(\mathcal{B})$;
2. $Gf(\mathcal{A} \cap \mathcal{B}) = Gf(\mathcal{A}) \cap Gf(\mathcal{B})$;
3. $Gf(\mathcal{A}^{\perp}) = Gf(\mathcal{A})^{\perp}$ *provided f is continuous and Y is a T_4-space.*

Proof. 1. The first item follows from the equalities

$$Gf(\mathcal{A}\cup\mathcal{B}) = f(\mathcal{A}\cup\mathcal{B})^{\perp\perp} = \big(f(\mathcal{A})\cup f(\mathcal{B})\big)^{\perp\perp} = \\ = \big(f(\mathcal{A})^{\perp}\cap f(\mathcal{B})^{\perp}\big)^{\perp} = \big(f(\mathcal{A})^{\perp\perp}\cup f(\mathcal{B})^{\perp\perp}\big) = Gf(\mathcal{A})\cup Gf(\mathcal{B}).$$

2. By analogy we can derive the second item from the equality $f(\mathcal{A}\cap\mathcal{B}) = f(\mathcal{A})\cap f(\mathcal{B})$ which can be proved as follows. Given any closed subset $E\in f(\mathcal{A})\cap f(\mathcal{B})$ of Y, find $A\in\mathcal{A}$ and $B\in\mathcal{B}$ with $A\subset f^{-1}(E)$ and $B\subset f^{-1}(E)$. Then $A\cup B\in\mathcal{A}\cap\mathcal{B}$ and $A\cup B\subset f^{-1}(E)$, which means that $E\in f(\mathcal{A}\cap\mathcal{B})$. This proves the inclusion $f(\mathcal{A})\cap f(\mathcal{B})\subset f(\mathcal{A}\cap\mathcal{B})$. The reverse inclusion is trivial.

3. Assume that the function f is continuous and Y is a T_4-space Y. We need to check that $Gf(\mathcal{A}^{\perp}) = Gf(\mathcal{A})^{\perp}$. Since $Gf(\mathcal{A})^{\perp} = f(\mathcal{A})^{\perp\perp\perp} = f(\mathcal{A})^{\perp}$, the inclusion

$$\overline{f(\mathcal{A}^{\perp})} = Gf(\mathcal{A}^{\perp}) \subset Gf(\mathcal{A})^{\perp} = f(\mathcal{A})^{\perp}$$

will follow as soon as we check that $f(\mathcal{A}^{\perp})\subset f(\mathcal{A})^{\perp}$. Take any set $A\in f(\mathcal{A}^{\perp})$ and find $B\in\mathcal{A}^{\perp}$ with $B\subset f^{-1}(A)$. Then for any set $C\in f(\mathcal{A})$ we can find a set $D\in\mathcal{A}$ with $D\subset f^{-1}(C)$. Since $B\in\mathcal{A}^{\perp}$, we get that

$$\varnothing\neq B\cap D\subset f^{-1}(A)\cap f^{-1}(C) = f^{-1}(A\cap C)$$

and hence $A\cap C\neq\varnothing$. We have shown that the set A meets each set $C\in f(\mathcal{A})$ and hence $A\in f(\mathcal{A})^{\perp}$, which proved the desired inclusion $\underline{f(\mathcal{A}^{\perp})}\subset f(\mathcal{A})^{\perp}$.

To prove the reverse inclusion $f(\mathcal{A})^{\perp}\subset\overline{f(\mathcal{A}^{\perp})}$ we shall need the continuity of f and the T_4-property of Y. Assuming that $f(\mathcal{A})^{\perp}\setminus\overline{f(\mathcal{A}^{\perp})}$ contains some closed set $F\subset Y$, we can find an open neighborhood $O(F)\subset Y$ of F, containing no set from the inclusion hyperspace $\overline{f(\mathcal{A}^{\perp})}$. By the T_4-axiom for Y, the closed set $C = Y\setminus O(F)$ of Y has a neighborhood $O(C)$ whose closure $E = \overline{O(C)}$ lies in $Y\setminus F$. By the continuity of f the set $f^{-1}(E)$ is closed in X. We claim that this set belongs to $\mathcal{A} = \mathcal{A}^{\perp\perp}$. Take any set $A\in\mathcal{A}^{\perp}$ and observe that the closure $\overline{f(A)}$ of its image $f(A)$ in Y belongs to $f(\mathcal{A}^{\perp})$. The choice of $O(F)$ guarantees that $\overline{f(A)}$ intersects $C = Y\setminus O(F)$ and consequently, $f(A)$ intersects $O(C)$ and $E = \overline{O(C)}$. Then A must intersect $f^{-1}(E)$, which means that $f^{-1}(E)\in\mathcal{A}^{\perp\perp} = \mathcal{A}$ and then $E\in f(\mathcal{A})$, which is impossible because E does not intersect the set $F\in f(\mathcal{A})^{\perp}$. □

The following example shows that the T_4-property of Y in the third item of Proposition 19 is essential.

Example 4. Consider the identity inclusion $f: X\to Y$ of the singleton $X = \{a\}$ into the connected three-point space $Y = (\{a,b,c\},\tau)$ endowed with the topology $\tau = \big\{\varnothing,\{a\},\{a,b\},\{a,c\},\{a,b,c\}\big\}$. The inclusion f induces the lattice homomorphism $Gf: G(X)\to G(Y)$. We claim that $Gf(\mathcal{F}^{\perp})\neq Gf(\mathcal{F})^{\perp}$ where $\mathcal{F} = \big\{\{a\}\big\}$ is the unique element of $G(X)$. Observe that

$$Gf(\mathcal{F}^{\perp}) = Gf(\mathcal{F}) = \overline{f(\mathcal{F})} = \big\{\{a,b,c\},\{b,c\}\big\}$$

while

$$Gf(\mathcal{F})^{\perp} = \big\{\{a,b,c\},\{b,c\}\big\}^{\perp} = \big\{\{a,b,c\},\{b,c\},\{b\},\{c\}\big\}\neq Gf(\mathcal{F}^{\perp}).$$

The continuity of the induced map Gf is a delicate question.

Proposition 20. *A function $f: X\to Y$ between T_1-spaces is continuous provided the function $Gf: G(X)\to G(Y)$ is continuous.*

Proof. Using the embedding $i_X : X \to G(X)$, $i_X : x \mapsto \uparrow x$, identify X with a subspace of $G(X)$. By analogy, identify Y with the subspace $i_Y(Y)$ of $\exp(Y)$. For every $x \in X$ consider the ultrafilters $\uparrow x = \{F \in \exp(X) : x \in F\}$ and $\uparrow f(x) = \{F \in \exp(Y) : f(x) \in F\}$. It follows from the definition of the map Gf that $Gf(\uparrow x) = \uparrow f(x)$. This can be written as $i_Y \circ f = Gf \circ i_X$. Since the maps i_X, i_Y are embeddings, the continuity of Gf implies the continuity of the map f. □

For normal spaces Y the previous proposition can be reversed.

Proposition 21. *If $f : X \to Y$ is a continuous function from a topological space X to a T_4-space Y, then the function $Gf : G(X) \to G(Y)$ is continuous.*

Proof. It suffices to check that for every open set $U \subset Y$ the preimages $Gf^{-1}(U^+)$ and $Gf^{-1}(U^-)$ are open in $G(X)$. Fix any inclusion hyperspace $\mathcal{F} \in Gf^{-1}(U^+)$ and let $\mathcal{E} = Gf(\mathcal{F})$. Since $\mathcal{E} \in U^+$, there is a set $E \in \mathcal{E}$ with $f(F) \subset E \subset U$ for some $F \in \mathcal{F}$. Since Y is a T_4-space, we can find an open subset $W \subset Y$ such that $E \subset W \subset \overline{W} \subset U$. Then $f^{-1}(W)^+$ is a neighborhood of $\mathcal{F}$ in $G(X)$ such that $Gf(\mathcal{F}') \in U^+$ for every $\mathcal{F}' \in f^{-1}(W)^+$. This witnesses that the set $Gf^{-1}(U^+)$ is open in $G(X)$.

To show that $Gf^{-1}(U^-)$ is open in $G(X)$, fix any inclusion hyperspace $\mathcal{F} \in Gf^{-1}(U^-)$ and let $\mathcal{E} = Gf(\mathcal{F})$. Since $X \setminus U \notin \mathcal{E}$, there is a neighborhood $W \subset Y$ of $X \setminus U$ containing no closed set $E \in \mathcal{E}$, which means that $E \cap (Y \setminus W) \neq \varnothing$ for all $E \in \mathcal{E}$. Since Y is a T_4-space, there is an open set $V \subset Y$ such that $Y \setminus W \subset V \subset \overline{V} \subset U$. Therefore, $\mathcal{E} \in V^-$. One can easily check that $\mathcal{F} \in f^{-1}(V)^-$ and $f^{-1}(V)^-$ is an open neighborhood of $\mathcal{F}$ lying in $Gf^{-1}(U^-)$ and witnessing that $Gf^{-1}(U^-)$ is an open set in $G(X)$. □

The following simple example shows that, in general, the continuity of Gf does not follow from the continuity of f.

Example 5. Consider the identity inclusion $f : X \to Y$ of the connected two-point space $X = \{a, b\}$ with the topology $\tau_X = \big\{\varnothing, \{a\}, \{a, b\}\big\}$ into the connected three-point space $Y = \{a, b, c\}$ endowed with the topology

$$\tau_Y = \big\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\big\}.$$

We claim that the embedding f induces a discontinuous function $Gf : G(X) \to G(Y)$.

Observe that the space $G(X)$ consists of two inclusion hyperspaces: $\mathcal{A} = \big\{\{a, b\}\big\}$ and $\mathcal{B} = \big\{\{b\}, \{a, b\}\big\}$. Writing down their images in $G(Y)$, we obtain:

$$Gf(\mathcal{A}) = \big\{\{b, c\}, \{a, b, c\}\big\} \text{ and } Gf(\mathcal{B}) = \big\{\{b\}, \{b, c\}, \{a, b, c\}\big\}.$$

Observe that the open subset $\{a, b\}^+$ of $G(Y)$ contains the inclusion hyperspace $Gf(\mathcal{B})$ but not $Gf(\mathcal{A})$. Assuming the continuity of the function Gf, we would conclude that $U = Gf^{-1}(\{a, b\}^+) = \{\mathcal{B}\}$ is an open set in $G(X)$. But this is not true because the unique non-trivial open set in $G(X)$ is $\{a\}^- = \{\mathcal{A}\}$.

Next, we study the natural question if the induced maps between spaces of incluion hyperpsaces preserve the composition of maps.

Proposition 22. *Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps between topological spaces. For any inclusion hyperspace $\mathcal{F} \in G(X)$ we get*

1. *$G(g\circ f)(\mathcal{F})\subset(Gg\circ Gf)(\mathcal{F}))$;*
2. *$G(g\circ f)(\mathcal{F})=(Gg\circ Gf)(\mathcal{F})$ provided Z is a T_4-space.*

Proof. 1. The first item will follow from the closedness of $(Gg\circ Gf)(\mathcal{F})=Gg(Gf(\mathcal{F}))$ as soon as we check that $g\circ f(\mathcal{F})\subset Gg(Gf(\mathcal{F}))$. Take any closed subset $A\in g\circ f(\mathcal{F})$. Then $g^{-1}(A)\in f(\mathcal{F})\subset\overline{f(\mathcal{F})}=Gf(\mathcal{F})$ and hence $A\in g(Gf(\mathcal{F}))\subset\overline{g(Gf(\mathcal{F}))}=Gg(Gf(\mathcal{F}))$.

2. Assuming that Z is a T_4-space, we shall prove that $(Gg\circ Gf)(\mathcal{F})\subset G(g\circ f)(\mathcal{F})$. Since $(Gg\circ Gf)(\mathcal{F})=\overline{g(Gf(\mathcal{F}))}$, it suffices to check that $g(Gf(\mathcal{F}))\subset G(g\circ f)(\mathcal{F})$. Take any closed set $A\in g(Gf(\mathcal{F}))$. Assuming that $A\notin G(g\circ f)(\mathcal{F})=\overline{g\circ f(\mathcal{F})}$ we may find an open neighborhood $O(A)\subset Z$ of A containing no set from the family $g\circ f(\mathcal{F})$. Since Z is a T_4-space, the open neighborhood $O(A)$ contains the closure of another open neighborhood $O_1(A)$ of A in Z. For every $F\in\mathcal{F}$ the closure $\overline{g(f(F))}$ belongs to $g\circ f(\mathcal{F})$ and intersects the closed set $Z\setminus O(A)$ by the choice of $O(A)$. Then $g(f(F))$ intersects the closed neighborhood $B=Z\setminus O_1(A)$ of $Z\setminus O(A)$. Consequently, $g^{-1}(B)$ meets each set $f(F)$, $F\in\mathcal{F}$. This implies that $g^{-1}(B)\in f(\mathcal{F})^{\perp}$. Since $g^{-1}(A)\in Gf(\mathcal{F})=f(\mathcal{F})^{\perp\perp}$, we conclude that $g^{-1}(A)$ meets $g^{-1}(B)$ which is a contradiction (because $A\cap B=\varnothing$). □

The following example shows that the last item of Proposition 22 is not true without the T_4-assumption.

Example 6. Let $X=\{a\}$ be a singleton and $Y=Z=\{a,b,c\}$ be a connected three-point space endowed with the topology

$$\tau=\big\{\varnothing,\{a\},\{a,b\},\{a,c\},\{a,b,c\}\big\}.$$

Let $f:X\to Y$, $f:a\mapsto a$, be the identity inclusion and $g:Y\to Z$ be the map defined by $g(a)=a$ and $g(b)=g(c)=b$. Let $\mathcal{F}=\big\{\{a\}\big\}$ be the unique inclusion hyperspace in $G(X)$. It is easy to see that

$$\begin{aligned}G(g\circ f)(\mathcal{F})&=\big\{\{b,c\},\{a,b,c\}\big\},\\ Gf(\mathcal{F})&=\big\{\{b,c\},\{a,b,c\}\big\},\\ Gg(Gf(\mathcal{F}))&=\big\{\{b\},\{b,c\},\{a,b,c\}\big\}\end{aligned}$$

and hence $G(g\circ f)(\mathcal{F})\neq Gg(Gf(\mathcal{F}))$.

For continuous maps into T_4-spaces the induced maps preserve some important subspaces of the spaces of inclusion hyperspaces.

Proposition 23. *Let $f:X\to Y$ be a continuous map from a topological space X to a T_4-space Y. Then*

1. *$Gf(N_k(X))\subset N_k(Y)$ for every $k\ge 2$;*
2. *$Gf(N_{<\omega}(X)\subset N_{<\omega}(Y)$;*
3. *$Gf(\lambda_2(X))\subset\lambda_2(Y)$;*
4. *$Gf(\mathrm{Fil}(X))\subset\mathrm{Fil}(Y)$.*

Proof. 1. Fix any k-linked inclusion hyperspace $\mathcal{F}\in N_k(X)$. Then the family $f(\mathcal{F})\subset\exp(Y)$ is k-linked as well. It remains to prove that its closure $\overline{f(\mathcal{F})}=Gf(\mathcal{F})$ is k-linked. Assuming the converse, we can find k sets $A_1,\dots,A_k\in\overline{f(\mathcal{F})}$ with empty intersection $A_1\cap\dots\cap A_k$. Repeating the argument of the proof of Proposition 13, we may enlarge each set A_i to an open set $U_i \subset Y$ so that still $U_1 \cap \cdots \cap U_k = \varnothing$. For every $i \le k$ the set $A_i \in \overline{f(\mathcal{F})}$ and consequently, the open neighborhood U_i^+ of A_i has non-empty intersection with the set $f(\mathcal{F})$. This allows us to find a closed subset $B_i \in f(\mathcal{F}) \cap U_i^+$. Since $B_i \subset U_i$ for all $i \le k$, the intersection $B_1 \cap \cdots \cap B_k$ is empty, contradicting to the fact that $f(\mathcal{F})$ is k-linked.

2. The second item follows immediately from the first one.

3. Assume that $\mathcal{F} \in \lambda_2(X)$ is a maximal linked inclusion hyperspace. Since $\mathcal{F}^\perp = \mathcal{F}$, we may apply Proposition 19(3) to conclude that $Gf(\mathcal{F})^\perp = Gf(\mathcal{F}^\perp) = Gf(\mathcal{F})$. The latter equality witnesses that $Gf(\mathcal{F})$ is a maximal linked inlcusion hyperpsace on Y.

4. Take any filter $\mathcal{F} \in \mathrm{Fil}(X)$ and consider its image $Gf(\mathcal{F})$. Assuming that $Gf(\mathcal{F}) \notin \mathrm{Fil}(Y)$, we may find two sets $A_1, A_2 \in Gf(\mathcal{F})$ with $A_1 \cap A_2 \notin Gf(\mathcal{F})$. Next, find an open neighborhood $U \subset Y$ of $A_1 \cap A_2$ such that U contains no set from $Gf(\mathcal{F})$. Repeating the proof of Proposition 12 we can use the T_4-property of Y to find open neighborhoods U_1 and U_2 of the sets A_1, A_2 such that $U_1 \cap U_2 \subset U$. Those neighborhoods induce neighborhoods U_1^+ and U_2^+ of A_1, A_2 in $G(Y)$ intersecting the set $f(\mathcal{F})$. So we can find a closed set $B_i \in U_i^+ \cap f(\mathcal{F})$ for $i = 1, 2$. The filter $\mathcal{F}$ contains the preimages $f^{-1}(B_1)$ and $f^{-1}(B_2)$ and conseuqntlyconsequently, contains their intersection $f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2)$, which means that $B_1 \cap B_2 \in f(\mathcal{F})$. Since $B_1 \cap B_2 \subset U_1 \cap U_2 \subset U$ we get a contradiction with the choice of the set U as a neighborhood of $A_1 \cap A_2$ containing no set from $\overline{f(\mathcal{F})} \supset f(\mathcal{F})$. □

Next, we search for conditions guaranteeing that the map Gf is injective. We define an injective continuous map $f : X \to Y$ between two topological spaces to be a *C^*-embedding* if for any closed disjoint sets $A, B \subset X$ their images $f(A)$ and $f(B)$ have disjoint closures in Y.

Proposition 24. *If $f : X \to Y$ is a C^*-embedding of a topological space X into a topological space Y, then the map $Gf : G(X) \to G(Y)$ is injective.*

Proof. Fix two distinct inclusion hyperspaces $\mathcal{F}, \mathcal{E} \in G(X)$. Then either $\mathcal{F} \not\subset \mathcal{E}$ or $\mathcal{E} \not\subset \mathcal{F}$. Without loss of generality, $\mathcal{F} \not\subset \mathcal{E}$, which means that $F \notin \mathcal{E}$ for some $F \in \mathcal{F}$ and there is a neighborhood $O(F) \subset X$ of F containing no set $E \in \mathcal{E}$. Since f is a C^*-embedding, the sets $A = \mathrm{cl}_Y(f(F))$ and $B = \mathrm{cl}_Y(f(X \setminus O(F)))$ are disjoint in Y. It follows from the definition of $Gf(\mathcal{F})$ that $A \in Gf(\mathcal{F})$. On the other hand, $A \notin Gf(\mathcal{E})$ because A has the neighborhood $Y \setminus B$ containing no set $f(E)$, $E \in \mathcal{E}$ (assuming conversely that $f(E) \subset Y \setminus B \subset Y \setminus f(X \setminus O(F))$ we would get $E \subset f^{-1}(Y \setminus f(X \setminus O(F))) = O(F)$ which contradicts the choice of $O(F)$). □

Proposition 25. *If $f : X \to Y$ is a map between T_1-spaces with dense image $f(X)$ in Y, then the image $Gf(G(X))$ is dense in $G(Y)$.*

Proof. Repeating the proof of Proposition 7 we can show that $Gf(G^\bullet(X))$ is dense in $G(Y)$ and this implies the density of $Gf(GX)$ in $G(Y)$. □

Corollary 4. *If $f : X \to Y$ is a C^*-embedding of a space X to a normal space Y with dense image $f(X)$ in Y, then the induced map $Gf : G(X) \to G(Y)$ is a homeomorphism.*

Proof. By Theorem 2 and Corollary 1, the spaces $G(X)$ and $G(Y)$ are Hausdorff compacta. Now Propositions 21–25 imply that map $Gf : G(X) \to G(Y)$ is continuous, injective, and has dense image. Finally, the compactness of $G(X)$ implies that Gf is a homeomorphism. □

Applying this corollary to the embedding of a normal space X into its Stone-Čech compactification βX we conclude that the space $G(X)$ is canonically homeomorphic to $G(\beta X)$. This observation is due to E.Moiseev [11].

Corollary 5. *Let X be a normal space and $f : X \to \beta X$ be the embedding of X into the Stone-Čech compactification of X. Then the map $Gf : G(X) \to G(\beta X)$ is a homeomorphism.*

A similar statement is true for some subspaces of $G(X)$.

Corollary 6. *Let X be a normal space and $f : X \to \beta X$ be the embedding of X into the Stone-Čech compactification of X. Then the restrictions*

$$Gf|\mathrm{Fil}(X) : \mathrm{Fil}(X) \to \mathrm{Fil}(\beta X), \quad Gf|N_{<\omega}(X) : N_{<\omega}(X) \to N_{<\omega}(\beta X),$$
$$Gf|\lambda(X) : \lambda(X) \to \lambda(\beta X), \quad Gf|N_k(X) : N_k(X) \to N_k(\beta X), \ k \geq 2,$$

are homeomorphisms.

Proof. We shall present the detail proofs for the spaces $N_k(X)$ and $\lambda(X)$. For other spaces the proofs are analogous.

Since $Gf : G(X) \to G(\beta X)$ is a homeomorphism, to prove that the restriction $Gf|N_k(X) : N_k(X) \to N_k(\beta X)$ is a homeomorphism, it suffices to check that $Gf(N_k(X)) = N_k(\beta X)$. By Proposition 23, $Gf(N_k(X)) \subset N_k(\beta X)$, and by Proposition 13, the spaces $N_k(X)$ and $N_k(\beta X)$ are closed in the compact Hausdorff spaces $G(X)$ and $G(\beta X)$.

Repeating the proof of Proposition 7 we can show that $Gf(N_k^\bullet(X))$ is dense in $N_k(\beta X)$ and this implies the density of $Gf(N_k(X))$ in $N_k(\beta X)$. This fact combined with the compactness of $Gf(N_k(X))$ implies the desired equality $Gf(N_k(X)) = N_k(\beta X)$.

To see that $Gf|\lambda(X) : \lambda(X) \to \lambda(\beta X)$ is a homeomorphism, observe that $\lambda(X) = N_2(X) \cap N_2(X)^\perp$ where $N_2(X)^\perp = \{\mathcal{F}^\perp : \mathcal{F} \in N_2(X)\}$. By Proposition 19(3), $Gf(\mathcal{F}^\perp) = Gf(\mathcal{F})^\perp$ for every $\mathcal{F} \in G(X)$. Thus we can write $Gf(\lambda(X)) = Gf(N_2(X) \cap N_2(X)^\perp) = Gf(N_2(X)) \cap Gf(N_2(X)^\perp) = Gf(N_2(X)) \cap Gf(N_2(X))^\perp = N_2(\beta X) \cap N_2(\beta X)^\perp = \lambda(\beta X)$. □

The structure of $G(X)$ over finite spaces. In this section we present some information of the space $G(X)$ and its subspaces for a finite space X. In this case the space $G(X)$ is finite. The problem of calculation the cardinality $G(n)$ of $G(X)$ for a space X of cardinality n is not trivial and is tightly connected with the classical (and still unsolved) problem of Dedekind [3] who suggested to determine the number $M(n)$ of all monotone Boolean functions of n Boolean variable. The function $M(n)$ growths very quickly. Its exact values are known only for $n \leq 8$ and are given in the following table taken from On-Line Encyclopedia of Integer Sequences [1].

n	$\lvert M(n)\rvert$
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788

www.research.att.com/cgi-bin/access.cgi/as/njas/sequences/eisA.cgi?Anum=000372

Observe that for each inclusion hyperspace $\mathcal{F} \in G(X)$ the characteristic function $\chi_{\mathcal{F}} : \mathcal{P}(X) \to \{0,1\}$ of $\mathcal{F}$ is monotone with respect to the inclusion relation on the power-set $\mathcal{P}(X)$ of X. Moreover, $\chi_{\mathcal{F}}(\varnothing) = 0$ and $\chi_F(X) = 1$. Conversely, each monotone function $f : \mathcal{P}(X) \to \{0,1\}$ with $f(\varnothing) = 0$ and $f(X) = 1$ determines an inclusion hyperspace $f^{-1}(1)$. This observation implies that for a finite set X of size n the size $G(n)$ of the set $G(X)$ equals $M(n) - 2$. In such a way we obtain the values of $G(n)$ for $n \le 8$. The cardinalities of other subsets of $G(X)$ for $|X| \le 6$ are given in the following table:

n	$G(n)$	$N_2(n)$	$\lambda_2(n)$	$N_3(n)$	$\lambda_3(n)$	$N_4(n)$	$N_{<\omega}(n)$	$\mathrm{Fil}(n)$
1	1	1	1	1	1	1	1	1
2	4	3	2	3	2	3	3	3
3	18	11	4	10	3	10	10	7
4	166	80	12	54	5	53	53	15
5	7579	2645	81	762	20	687	686	31
6	7828352	?	2646	?	?	?	43285	63

Some of entries of the table were found by computer calculations. On the other hand, for $\mathrm{Fil}(n)$, $N_{<\omega}(n)$, and $\lambda_2(n)$ there are recurrent formulas.

Proposition 26. *For any $n \in \mathbb{N}$ the following formulas hold:*

1. $\mathrm{Fil}(n) = 2^n - 1$;
2. $N_{<\omega}(n) = \sum_{k=1}^n C_n^k(1 + G(n-k) - N_{<\omega}(n-k))$;
3. $N_k(n) = N_{<\omega}(n)$ *for* $k \ge n$;
4. $N_{n-1}(n) = 1 + N_{<\omega}(n)$.
5. $\lambda_2(n) = 1 + N_2(n-1)$.

Proof. Let X be a set of cardinality $|X| = n$.

1. Since for every filter $\mathcal{F}$ on X the intersection $\cap\mathcal{F}$ is a non-empty subset generating $\mathcal{F}$, the number of filters equals the number of non-empty subsets of X, which is equal to $2^n - 1$.

2. It follows that for each centered inclusion hyperspace $\mathcal{F}$ on X the intersection $\cap\mathcal{F}$ is not empty. Fix any non-empty subset $M = \{x_1, \dots, x_k\}$ of X and look at the centered families $\mathcal{F}$ with $\cap\mathcal{F} = M$. The largest among them is generated by M. All the other are of the form $\mathcal{F} = \{M \cup A : A \in \mathcal{A}\}$ where $\mathcal{A}$ is an inclusion hyperspace on $X \setminus M$ with $\cap\mathcal{A} = \varnothing$. The number of such inclusion hyperspaces $\mathcal{A}$ equals $G(n-k) - N_{<\omega}(n-k)$. Thus the number of all inclusion hyperspaces $\mathcal{F}$ with $\cap\mathcal{F} = M$ is equal to $1 + G(n-k) - N_{<\omega}(n-k)$. Since the number of k-element subsets of X equals C_n^k, we conclude that $N_{<\omega}(n) = \sum_{k=1}^n C_n^k(1 + G(n-k) - N_{<\omega}(n-k))$.

3. Assume that for some $k \ge n$ there exists a k-linked inclusion hyperspace $\mathcal{F}$ on set X, which is not $(k+1)$-linked. So, there exist $F_1, F_2, \dots F_{k+1} \in \mathcal{F}$ with empty intersection $F_1 \cap F_2 \cap \ldots \cap F_{k+1}$. Consequently, $X = (X \setminus F_1) \cup \ldots \cup (X \setminus F_{k+1})$. Since $|X| = n$, $X = \bigcup_{i \in I} X \setminus F_i$ for some set $I \subset \{1, \dots, k+1\}$ of size $|I| \le n$. Then $\bigcap_{i \in I} F_i = \varnothing$ contradicting the k-linkedness of $\mathcal{F}$. Thus each n-linked inclusion hyperspace on X is centered which yields $N_n(n) = N_{<\omega}(n)$.

4. The equality $N_{n-1}(n) = 1 + N_n(n) = 1 + N_{<\omega}(n)$ will follow as soon as we show that each $(n-1)$-linked not n-linked inclusion hyperspace $\mathcal{F}$ on X is equal to the inclusion hyperspace $\mathcal{L} = \{L \subset X : |L| \ge n-1\}$. First we check that $\mathcal{F} \subset \mathcal{L}$. Assuming that $\mathcal{F} \not\subset \mathcal{L}$, find a subset $F \in \mathcal{F}$ with $|F| \le n-2$. Then the inclusion hyperspace $\mathcal{F}' = \{F \cap A : A \in \mathcal{A}\}$

on F is $(n-2)$-linked and hence centered by the preceding item. Consequently, the inclusion hyperspace $\mathcal{F}$ is centered too, contradicting the fact that $\mathcal{F}$ is not n-linked. Therefore $\mathcal{F} \subset \mathcal{L}$, which means that $|F| \geq n-1$ for all $F \in \mathcal{F}$.

Being not n-linked, the inclusion hyperspace $\mathcal{F}$ contains subsets $F_1, F_2, \dots F_n \in \mathcal{F}$ with empty intersection $F_1 \cap F_2 \cap \dots \cap F_n$. It follows from $|X \setminus F_i| \leq 1$, $i \leq n$, and $X = (X \setminus F_1) \cup \dots \cup (X \setminus F_n)$ that $\{X \setminus F_i : i \leq n\} = \{A \subset X : |A| = 1\}$. Then $\mathcal{L} \subset \{X, F_i : i \leq n\} \subset \mathcal{F}$ and hence $\mathcal{F} = \mathcal{L}$.

5. Fix any point $x \in X$ and take any maximal linked inclusion hyperspace $\mathcal{L}$. If $\mathcal{L}$ is not an ultrafilter generated by x, then the family $\mathcal{L}_0 = \{A \in \mathcal{L} \colon A \subset X \setminus \{x\}\}$ is a linked inclusion hyperspace on $X \setminus \{x\}$. The maximality of $\mathcal{L}$ implies that $\mathcal{L} = \mathcal{L}_0 \cup \{A \subset X \colon x \in A,\ A \setminus \{x\} \in \mathcal{L}_0^{\perp}\}$. Consequently, the number of maximal linked inclusion hyperspaces is 1 plus the number of linked inclusion hyperspaces on $X \setminus \{x\}$ which can be written as $\lambda(n) = 1 + N_2(n-1)$. □

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