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У збірнику містяться матеріали Сьомої міжнародної алгебраїчної конференції в Україні, присвяченої 120-річчю від дня народження професора Харківського університету Антона Казиміровича Сушкевича.

The collection contains the abstracts of talks of 7^{th} International Algebraic Conference in Ukraine dedicated to the 120th anniversary of Professor of Kharkiv University Anton Kazimirovich Sushkevich.

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Матеріали подаються в авторській редакції. Відповідальність за достовірність інформації, коректність математичних викладок несуть автори. Тези доповідей опубліковано мовою оригіналу.

Посилання на матеріали збірника обов'язкові.

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Contents

Abstracts of Talks

On invariants of irreducible polynomials over a Henselian valued field

Kamal Aghigh

Let v be a henselian valuation of any rank of a field K and $\bar{v}$ be the extension of v to a fixed algebraic closure $\overline{K}$ of K. In 1970 James Ax proved that if α in $\overline{K}\backslash K$, is such that $[K(\alpha):K]$ is not divisible by the characteristic of the residue field of v then there exist a in K such that $\bar{v}(\alpha - a) \geq \Delta_K(\alpha)$, where

$$\Delta_K(\alpha) = min\{\bar{v}(\alpha-\alpha')|\ \alpha' \neq \alpha \text{ runs over } K\text{-conjugates of } \alpha\}.$$

In 1998, Khanduja generalized the above result by showing that (K, v) is a tame field, if and only if, to every $\alpha \in \overline{K}\backslash K$, there corresponds $a \in K$ such that $\bar{v}(\alpha - a) \geq \Delta_K(\alpha)$. The last statement is equivalent to saying that to every α in $\overline{K}\backslash K$, there corresponds β in $\overline{K}$ such that $[K(\beta) : K] < [K(\alpha) : K]$ and $\bar{v}(\alpha - \beta) \geq \Delta_K(\alpha)$. This has motivated us to consider similar problems for the constants $\omega_K(\alpha)$ and $\delta_K(\alpha)$ defined by

$$\omega_K(\alpha) = max\{\bar{v}(\alpha-\alpha')|\ \alpha' \neq \alpha \text{ runs over } K\text{-conjugates of } \alpha\}$$

and

$$\delta_K(\alpha) = sup\{\bar{v}(\alpha-\beta):\ \beta \in \overline{K}, [K(\beta):K] < [K(\alpha):K]\}.$$

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p^{th} roots of elements in finite groups

K. Ahmadidelir†, H. Doostie, A. Sadeghieh

The probability that a randomly chosen element in a finite group has a square root, has been investigated by certain authors. In this talk, we generalize this probability to p^{th} root ($p > 2$ is a prime number) and give some bounds for it. Also, if we denote this probability by $P_p(G)$ for a finite group G, we show that the set $\{P_p(G) \mid G \text{ is a finite group }\}$ is a dense subset of the closed interval $[0, 1]$, investigating this bounds for $P_p(G)$ promote us to pose an open problem concerning the rational subset of $[0, 1]$.

Our main results are:

Theorem 1. *Let G be a finite group of order divisible by prime p, where p is the smallest prime divisor of $|G|$. If the Sylow p–subgroup of G is not a proper normal elementary abelian subgroup of G, then*

$$P_p(G) \leqslant 1 - \frac{1}{\sqrt{|G|}}.$$

Theorem 2. *The set $\{P_p(G) \mid G \text{ is a finite group}\}$ is dense in* $[0, 1]$.

Again, as stated in [1] for $P_2(G)$, the following question is still open:

Open problem. *Which rational numbers in the interval* $[0, 1]$ *does the function $P_p(G)$ takes, as G runs over through the set of all finite groups?*

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Completion of pseudonormed rings and semi-isometric isomorphism

S. A. Aleschenko, V. I. Arnautov‡

Definition 1. *Let (R,ξ) and (R',ξ') be pseudonormed rings. A ring isomorphism $\varphi : R \to R'$ is called a semi-isometric isomorphism (see [1]) if there exists a pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$ such that:*

1. (R,ξ) is an ideal in $\left(\hat{R},\hat{\xi}\right)$;

2. φ can be extended up to an isometric homomorphism $\hat{\varphi} : \left(\hat{R},\hat{\xi}\right) \to (R',\xi')$.

Definition 2. *An element $a \in (R,\xi)$ of a pseudonormed ring (R,ξ) is called a generalized divisor of zero if there exists a subset $M \subseteq R$ such that:*

1. $\inf\{\xi(r) | r \in M\} > 0$*;*

2. $\inf\{a \cdot r | r \in M\} = 0$ *and* $\inf\{r \cdot a | r \in M\} = 0$.

Remark 1. *It's well known that for any pseudonormed ring (R,ξ) there exists its completion, i.e. there exists a pseudonormed ring $\widetilde{(R,\xi)}$ such that (R,ξ) is a dense subring in $\widetilde{(R,\xi)}$ and any Cauchy sequence of the pseudonormed ring (R,ξ) has its limit in $\widetilde{(R,\xi)}$.*

Theorem 1. *Let (R,ξ) and (R',ξ') be pseudonormed rings, and $\widetilde{(R,\xi)}$ and $\widetilde{(R',\xi')}$ be their completions, respectively. If $\varphi : (R,\xi) \to (R',\xi')$ is a semi-isometric isomorphism and the pseudonormed ring (R,ξ) has an element which isn't a generalized divisor of zero in (R,ξ) then φ can be extended up to a semi-isometric isomorphism $\tilde{\varphi} : \widetilde{(R,\xi)} \to \widetilde{(R',\xi')}$.*

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On the elementary obstruction in linear algebraic groups over pseudoglobal fields

V. Andriychuk

Let k be an algebraic function field in one variable with pseudofinite [1] constant field. It is known [2] that over k the group of R-equivalence classes and the defect of weak approximation are trivial for the following classes of linear algebraic groups over k: simply connected; adjoint; absolutely almost simple algebraic groups; and for inner forms of groups which are split by a metacyclic extension. In the case of an arbitrary connected group they are finite. Moreover, for a connected algebraic group the obstruction to the Hasse principle is a finite abelian group, and it is trivial for a simply connected group. The proofs of these properties are based on the fact that k is a good field, that is: (i) its cohomological dimension is 2, (ii) over any finite field extension K/k, for any central simple algebra A/K, the index of A and the exponent of A in $\operatorname{Br} K$ coincide; (iii) for any semisimple simply connected group G/k we have $H^1(k, G) = 0$.

Let X be a geometrically integral variety over k, $\bar{k}$ an algebraic closure of k, $\mathfrak{g}$ the absolute Galois group of k, $\bar{k}(X)$ function field of $\bar{X} = X \times_k \bar{k}$. The elementary obstruction is the class $ob(X) \in \operatorname{Ext}^1_{\mathfrak{g}}(\bar{k}(X)^*/\bar{k}^*, \bar{k}^*)$ of the extension of Galois modules $1 \to \bar{k}^* \to \bar{k}(X)^* \to \bar{k}(X)^*/\bar{k}^* \to 1$. It is shown in [3] that the properties (i), (ii), and (iii) allow us to prove the following result.

Theorem 1. *Let k be be an algebraic function field in one variable with pseudofinite [1] constant field of characteristic zero. Let X/k be a homogeneous space of a connected linear group G with connected geometric stabilizers. Then $X(k) \neq \emptyset$ if and only if $ob(X) = 0$.*

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Groups generated by slowmoving automata transformations

A. Antonenko*, E. Berkovich♭

We consider finite automata over a finite alphabet $X = \{0, 1, \ldots, m-1\}$ of input and output symbols. We denote the set of states by Q, the transition function by $\pi : X \times Q \to Q$ and the output function by $\lambda : X \times Q \to X$. Define $\lambda_q(x) = \lambda(x, q)$, $\lambda_q : X \to X$.

We consider transformations defined by automata with the following three properties: 1) for each state $q \in Q$ of an automaton, there exist at most one symbol $x \in X$ such that $\pi(x, q) \neq q$ (slowmoving automata); 2) there are no cycles except of loops in the Moore diagram of an automaton (automata of finite type); 3) for each state output function λ_q is circular shift : $\lambda_q = \sigma^k$, where $\sigma(i) = (i + 1)$ for $i = \overline{0, m-2}$, $\sigma(m-1) = 0$, k is an integer.

All such transformations for fixed m-symbol alphabet X generate the selfsimilar group which we denote by $GSlC_m$. We show that there is an irreducible basis $\left\{\alpha_i | i = \overline{0, \infty}\right\}$ of $GSlC_m$, where $\alpha_0 = (\alpha_0, \ldots, \alpha_0)\,\sigma$, $\alpha_i = (\alpha_{i-1}, \alpha_i, \ldots, \alpha_i)$, $i = \overline{1, \infty}$ are transformations of m-symbol alphabet X [1]. Define also $GSlC_m(k) = \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \rangle$ which are also selfsimilar. It is easy to see that $GSlC_2(2) = D_\infty$ is the infinite dihedral group. Group $GSlC_2(3)$ was considered in [2] as group $G_{1324} \simeq G_{937}$. In [2] it is proved that G_{937} is fractal, regular weakly branch over its commutant, non contracting and has exponential growth. We prove that $GSlC_m$ and $GSlC_m(k)$ are fractal, non contracting and have exponential growth for any $m \geq 2$, $k \geq 2$. If a word over $\left\{\alpha_i | i = \overline{0, \infty}\right\}$ is a relation then the number of α_i in it for any given i is divisible by m. The set of all such words is the commutant of $GSlC_m$.

We find the algorithm of constructing of non-trivial relations in $GSlC_m$ and $GSlC_m(k)$. Taking a word with some identity projections it increases the number of identity projections.

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Prime spectrum and primitive Leavitt path algebras

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Leavitt path algebras of row-finite graphs have been recently introduced in [1] and [2]. They have become a subject of significant interest, both for algebraists and for analysts working in C*-algebras. The Cuntz-Krieger algebras $C^*(E)$ (the C*-algebra counterpart of these Leavitt path algebras) are described in [5].

For a field K, the algebras $L_K(E)$ are natural generalizations of the algebras investigated by Leavitt in [4], and are a specific type of path K-algebras associated to a graph E (modulo certain relations). The family of algebras which can be realized as the Leavitt path algebras of a graph includes matrix rings

$\mathbb{M}_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $\mathbb{M}_\infty(K)$ denotes matrices of countable size with only a finite number of nonzero entries), the Toeplitz algebra, the Laurent polynomial ring $K[x, x^{-1}]$, and the classical Leavitt algebras $L(1, n)$ for $n \geq 2$ (the latter being universal algebras without the Invariant Basis Number condition).

In this work we determine the prime and primitive Leavitt path algebras. The main inspiration springs out of the complete description of the primitive spectrum of a graph C*-algebra $C^*(E)$ carried out by Hong and Szymański in [6]. Concretely, in [6, Corollary 2.12], the authors found a bijection between the set $\mathrm{Prim}(C^*(E))$ of primitive ideals of $C^*(E)$ and some sets involving maximal tails and points of the torus $\mathbb{T}$. We give the algebraic version of this by exhibiting a bijection between the set of prime ideals of $L_K(E)$, and the set formed by the disjoint union of the maximal tails of the graph $\mathcal{M}(E)$ and the cartesian product of maximal tails for which every cycle has an exit $\mathcal{M}_\tau(E)$ and the nonzero prime ideals of the Laurent polynomial ring $\mathrm{Spec}(K[x, x^{-1}])^*$.

In addition, the primitive Leavitt path algebras are characterized. Concretely, $L_K(E)$ is left primitive if and only if $L_K(E)$ is right primitive if and only if every cycle in the graph E has an exit and $E^0 \in \mathcal{M}(E)$.

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About metrizable, linear, group topologies of the Abelian groups, which coincide on some subgroup and on some factor group[1]

V. I. Arnautov

Notations. During all this work, if it is not stipulated opposite, we shall adhere to the following notations;

1. p is some fixed prime number;
2. n is some fixed natural number;
3. G is an Abelian group;
4. G_1 and G_2 are subgroups of the group G;
5. If τ is a group topology on G, then we denote by $\tau|_{G_1}$ the induced topology on G_1, i.e. $\tau|_{G_1} = \{U \bigcap G_1 | U \in \tau\}$;
6. $\omega : G \to G/G_2$ is the natural homomorphism (i.e. $\omega(g) = g + G_2$ for any $g \in G$);
7. If (G, τ) is a topological group, then we denote by $(G, \tau)/G_2$ the topological group $(G/G_2, \bar{\tau})$, where $\bar{\tau} = \{\omega(U) | U \in \tau\}$.

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Proposition 1. *Let τ and τ' be group topologies on a group G and G_1 and G_2 be such subgroups of the group G, that $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$. If $\tau|_{G_1} = \tau'|_{G_1}$ and $(G, \tau)/G_2 = (G, \tau')/G_2$, then the topological groups (G, τ) and (G, τ') possess such bases $\{U_\gamma | \gamma \in \Gamma\}$ and $\{U'_\gamma | \gamma \in \Gamma\}$ of the neighborhoods of zero respectively, that $U_\gamma \bigcap G_1 = U'_\gamma \bigcap G_1$ and $G_2 + U_\gamma = G_2 + U'_\gamma$ for any $\gamma \in \Gamma$, and if topologies τ and τ' are linear, then U_γ and U'_γ are subgroups of group G.*

Theorem 1. *Let G be any Abelian group of the period p^n and $G_2 = \{g \in G | p \cdot g = 0\}$. Let τ and τ' be such metrizable, linear, group topologies that the subgroup $G_1 = \{g \in G | p^{n-1} \cdot g = 0\}$ is a closed subgroup in each of topological groups (G, τ) and (G, τ'). Then $\tau|_{G_1} = \tau'|_{G_1}$ and $(G, \tau)/G_2 = (G, \tau')/G_2$ if and only if there exists such group isomorphism $\varphi : G \to G$, that the following conditions are satisfied:*

1. *$\varphi(G_1) = G_1$;*
2. *$g - \varphi(g) \in G_2$ for any $g \in G$;*
3. *$\varphi : (G, \tau) \to (G, \tau')$ is a topological isomorphism (i.e. open and continuous isomorphism).*

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Radical rings with center of finite index

O. D. Artemovych

In [1] (see also [2]) F. Szasz asked: "Es sei

$$\hat{a} = \{(1-x)a(1-x)^{-1} | x \in A\}$$

in einem Jacobsonschen Radikalring A. Wann ist jede Klasse $\hat{a}$ endlich, und wann ist die Anzahl der Klassen $\hat{a}$ endlich? (Problem 88)".

In this way we prove

Theorem 1. *Let R be a Jacobson radical ring. Then its adjoint group R° is an FC-group if and only if the additive group $Z(R)^+$ of the center $Z(R)$ has a finite index in the additive group R^+ of a ring R.*

This gives an answer on the first part of Problem 88. In [3] we prove that a Jacobson radical ring R with the torsion adjoint group R° which has finitely many conjugacy classes is finite. This is partial answer on the second part of Problem 88.

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On saturated subfields of invariants of finite groups

I. V. Arzhantsev†, A. P. Petravchuk‡

Consider the field $k(x_1, \ldots, x_n)$ of rational functions over an arbitrary field k. A rational function ψ is called *closed* if the subfield $k(\psi)$ is algebraically closed in the field $k(x_1, \ldots, x_n)$. For any $\phi \in k(x_1, \ldots, x_n) \setminus k$ there exist a closed function ψ and an element $H(t) \in k(t)$ such that $H(\psi) = \phi$. The element ψ is called *generative* for the element ϕ. Generative rational functions in two variables appeared in classical works of H.Poincare on algebraic integration of differential equations, the general case is considered in [3]. Closed rational functions were studied by many authors, see for example, [3] and [2].

A subfield $L \subseteq k(x_1, \ldots, x_n)$ will be called *saturated* if for any $\phi \in L \setminus k$ the generative rational function ψ of ϕ is contained in L. It is clear that every algebraically closed subfield in $k(x_1, \ldots, x_n)$ is saturated. The results below show that the converse is not true.

Theorem 1. *Let k be a field of characteristic zero and G be a finite subgroup of the automorphism group of $k(x_1, \ldots, x_n)$. Suppose that the group G has neither nontrivial abelian quotient groups nor quotient groups which are isomorphic to the alternating group A_5. Then the subfield $k(x_1, \ldots, x_n)^G$ is saturated in $k(x_1, \ldots, x_n)$.*

Note that the group $\mathrm{PSL}_2(\mathbb{C})$ contains a unique up to conjugation subgroup which is isomorphic to A_5. Denote its preimage in $\mathrm{SL}_2(\mathbb{C})$ by I_{120}. This is a group of order 120.

Theorem 2. *Let k be an algebraically closed field of characteristic zero and G be a finite subgroup of $\mathrm{GL_n(k)}$. The following conditions are equivalent:*

(i) the group G has neither nontrivial abelian quotient groups nor quotient groups isomorphic to I_{120};

(ii) the subfield $k(x_1, \ldots, x_n)^G$ is saturated in $k(x_1, \ldots, x_n)$.

Note that saturated subalgebras of the polynomial algebra $k[x_1, \ldots, x_n]$ were studied in [1].

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Isomorphisms of the groups $GL_n(R)$, $n \geqslant 4$ over associated graded rings

A. Atkarskaya

I.Z. Golubchik and A.V. Mikhalev in [1] and E.I. Zelmanov in [2] described isomorphisms of general linear groups $GL_n(R)$ over associative rings with $\frac{1}{2}$ for $n \geqslant 3$. Then I.Z. Golubchik in [3] described isomorphisms of groups $GL_n(R)$ over arbitrary associative rings, $n \geqslant 3$.

We extend Golubchik theorem for the case of an arbitrary associative graded ring R:

Definition 1. *Let* $R = \bigoplus_{g\in G} R_g$, $S = \bigoplus_{g\in G} S_g$ *be associative graded rings with* 1, $M_n(R)$, $M_n(S)$ *be graded matrix rings. Group isomorphism* $\varphi : GL_n(R) \longrightarrow GL_m(S)$ *is called an isomorphism respecting grading, if* $\varphi(GL_n(R) \cap M_n(R)_e) \subseteq GL_m(S)_e$ *and if* $A - E \in M_n(R)_g$, *then* $\varphi(A) - E \in M_m(S)_g$.

We prove the following theorem:

Theorem 1. *Let* G *be a commutative group,* $R = \bigoplus_{g\in G} R_g$, $S = \bigoplus_{g\in G} S_g$ *associative graded rings with unit,* $M_n(R)$, $M_m(S)$ *graded matrix rings,* $n \geqslant 4$, $m \geqslant 4$, *and* $\varphi : GL_n(R) \longrightarrow GL_m(S)$ *be a group isomorphism, respecting grading. Suppose that the isomorphism* φ^{-1} *also respects grading. Then there exist central idempotents* e *and* f *of the rings* $M_n(R)$ *and* $M_m(S)$ *respectively,* $e \in M_n(R)_0, f \in M_m(S)_0$, *a ring isomorphism* $\theta_1 : eM_n(R) \longrightarrow fM_m(S)$ *and a ring antiisomorphism* $\theta_2 : (1-e)M_n(R) \longrightarrow (1-f)M_m(S)$, *both of them preserve grading, such that* $\varphi(A) = \theta_1(eA) + \theta_2((1-e)A^{-1})$ *for all* $A \in GE_n(R)$ *(the subgroups generated by elementary and diagonal matrices).*

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The Rees-Suschkewitsch Theorem for simple topological semigroups

Taras Banakh†, Svetlana Dimitrova‡ and Oleg Gutik♯

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 2, 3].

We recall that a topological space X is

- *countably compact* if each closed discrete subspace of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *sequentially compact* if each sequence $\{x_n\}_{n\in\omega} \subset X$ has a convergent subsequence;
- *p-compact* for some free ultrafilter p on ω if each sequence $\{x_n\}_{n\in\omega} \subset X$ has a p-limit $x_\infty = \lim_{n\to p} x_n$ in X.

Here the notation $x_\infty = \lim_{n\to p} x_n$ means that for each neighborhood $O(x_\infty) \subset X$ of x_∞ the set $\{n \in \omega : x_n \in O(x_\infty)\}$ belongs to the ultrafilter p.

By [4], a topological space X is p-compact for some free ultrafilter p on ω if and only if each power X^κ of X is countably compact if and only if the power $X^{2^{\mathfrak{c}}}$ is countably compact. It is easy to see that each sequence $(x_n)_{n\in\omega}$ in a countably compact topological space X has p-limit $\lim_{n\to p} x_n$ for some free ultrafilter p on ω. We shall say that for some free filter p on ω a double sequence $\{x_{m,n}\}_{m,n\in\omega} \subset X$ has a double p-limit $\lim_{n\to p}\lim_{m\to p} x_{m,n}$ if $P = \{n \in \omega : \exists \lim_{m\to p} x_{m,n} \in X\} \in p$ and the sequence $(\lim_{m\to p} x_{m,n})_{n\in P}$ has a p-limit in X. We define a topological space X to be *doubly countably compact* if each double sequence

$(x_{m,n})_{m,n\in\omega}$ in X has a double p-limit $\lim\limits_{n\to p}\lim\limits_{m\to p} x_{m,n}\in X$ for some free ultrafilter p on ω. It can be shown that a topological space X is doubly countably compact if X is seqeuntially compact or p-compact for some free ultrafilter p.

Topological semigroups containing an idempotent can be characterized as follows.

Theorem 1. *A topological semigroup S contains an idempotent if and only if for some $x\in S$ the double sequence $(x^{m-n})_{m\geq n}$ has a double p-limit $\lim\limits_{n\to p}\lim\limits_{m\to p} x^{m-n}\in S$ for some free ultrafilter p on ω.*

Theorem 2. *A topological semigroup S contains an idempotent if S satisfies one of the following conditions:*

(*i*) *S is doubly countably compact;*

(*ii*) *S is sequentially compact;*

(*iii*) *S is p-compact for some free ultrafilter p on ω;*

(*iv*) *$S^{2^{\mathfrak{c}}}$ is countably compact;*

(*v*) *S^{κ^ω} is countably compact, where κ is the minimal cardinality of a closed subsemigroup of S.*

A semigroup S is called *simple* if S does not contain proper ideals. A simple semigroup with a primitive idempotent is called *completely simple*. Let X and Y be non-empty sets, G a semigroup and $\sigma\colon Y\times X\to G$ a map. Then the set $X\times G\times Y$ with the semigroup operation $(x,g,y)\cdot(x',g',y')=(x,g\sigma(y,x')g',y')$ is called the *Rees product* of the semigroup G over the sets X and Y with the sandwich function σ [5] and denoted by $[X,G,Y]_\sigma$. If G is a group then the semigroup $[X,G,Y]_\sigma$ is called a *paragroup* [1]. For any group G the semigroup $[X,G,Y]_\sigma$ is completely simple and every completely simple semigroup is isomorphic to a paragroup $[X,G,Y]_\sigma$ for some sets X and Y and a group G [5]. If G is a topological group, X and Y are Hausdorff topological spaces, and σ is a continuous map, then $[X,G,Y]_\sigma$ with a product topology is a topological semigroup called a *topological paragroup*. Every compact simple topological semigroup is topologically isomorphic to a topological paragroup $[X,G,Y]_\sigma$ for some Hausdorff compact topological spaces X and Y and compact topological group G [7]. We observe that the such theorem for finite semigroup was proved by A. K. Suschkewitsch in [6].

Theorem 3. *A topological semigroup S with countably compact square $S\times S$ is a topological paragroup if and only if S is simple and contains an idempotent.*

Theorem 4. *A topological semigroup S with pseudocompact square $S\times S$ is a topological paragroup if and only if S is simple and contains an idempotent.*

Theorem 5. *A pseudocompact topological semigroup S is a topological paragroup if and only if S is completely simple.*

We recall that a topological space X is called *sequential* if for each non-closed subset $A\subset X$ there is a sequence $\{a_n\}_{n\in\omega}\subset A$ that converges to some point $x\in X\setminus A$.

Theorem 6. *For a simple topological semigroup S the following conditions are equivalent:*

(*i*) *S is a regular sequential countably compact topological space;*

(*ii*) *S is topologically isomorphic to a topological paragroup $[X,G,Y]_\sigma$ for some regular sequential countably compact topological spaces X, Y and a sequential countably compact topological group G.*

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Twisted group algebras of semi-wild representation type

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We determine twisted group algebras of semi-wild representation type in the sense of Drozd definition [1]. We also introduce the concept of projective K-representation type for a finite group (tame, semi-wild, purely semi-wild) and we single out finite group of each type.

Let G be a finite group, G_p a Sylow p-subgroup of G, G_p' the commutant of G_p and C_p a Sylow p-subgroup of the commutant G' of the group G. We assume that $C_p \subset G_p$. Denote by D the subgroup of G_p such that $G_p' \subset D$ and $D/G_p' = \mathrm{soc}(G_p/G_p')$. Let K be a field of characteristic p and

$$i(K) = \begin{cases} t & \text{if} \quad [K:K^p] = p^t, \\ \infty & \text{if} \quad [K:K^p] = \infty. \end{cases}$$

Moreover, let K^* be the multiplicative group of K, $Z^2(G, K^*)$ the group of all K^*-valued normalized 2-cocycles of G, where we assume that G acts trivially on K^*, and $K^\lambda G$ be the twisted group algebra of the group G over the field K with a 2-cocycle $\lambda \in Z^2(G, K^*)$.

Theorem 1. *Let $p \neq 2$, G be a finite group, $\lambda \in Z^2(G, K^*)$ and $d = \dim_K \left(K^\lambda G_p / \mathrm{rad}\, K^\lambda G_p \right)$. Suppose that if $|G_p'| = p$, $pd = |G_p : G_p'|$ then D is abelian. The algebra $K^\lambda G$ is of semi-wild representation type if and only if the subalgebra $K^\lambda G_p$ is not uniserial.*

We say that a finite group G is of purely semi-wild projective K-representation type if $K^\lambda G$ is of semi-wild representation type for any $\lambda \in Z^2(G, K^*)$.

Theorem 2. *Let $p \neq 2$, G be a finite group and s the number of invariants of G_p/G_p'. Assume that if $|C_p| = p$, $s = i(K) + 1$ and D is non-abelian, then* $\exp D = p^2$. *A group G is of purely semi-wild projective K-representation type if and only if one of the following conditions is satisfied:*

(i) C_p is a non-cyclic group;

(ii) $s \geq i(K) + 2$;

(iii) $s = i(K) + 1$, $C_p = G'_p = \langle c \rangle$, $|C_p| \geq p^2$ *and* $g^p \in \langle c^p \rangle$ *for every* $g \in D$*;*

(iv) $s = i(K) + 1$, $C_p = G'_p$, $|C_p| = p$ *and* D *is an elementary abelian* p*-group.*

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Similarity transformation of the pair of matrices to the best partitioned-triangular form

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The problem of the transformation of matrices to maximum possible number of blocks has been solved. The method of commutative matrix and the method of invariant subspace [1, 2] have been used. Using these methods, the transformation in a finite number of steps has been done.

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The Hilbert series of the algebras of joint invariants and covariants of two binary forms

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Let V_{d_1}, V_{d_2} be the vector $\mathbb{C}$-spaces of the binary forms of degrees d_1 and d_2 endowed with the natural action of the group $SL_2(\mathbb{C})$. Consider the induced action of the group $SL_2(\mathbb{C})$ on the algebras of the polynomial functions $\mathcal{O}(V_{d_1} \oplus V_{d_2})$ and $\mathcal{O}(V_{d_1} \oplus V_{d_2} \oplus \mathbb{C}^2)$. The algebras

$$\mathcal{I}_{d_1,d_2} := \mathcal{O}(V_{d_1} \oplus V_{d_2})^{SL_2(\mathbb{C})} \text{ and } \mathcal{C}_{d_1,d_2} := \mathcal{O}(V_{d_1} \oplus V_{d_2} \oplus \mathbb{C}^2)^{SL_2(\mathbb{C})}$$

are called the algebra of simultaneous invariants and the algebra of simultaneous covariants for the binary forms. The reductivity of $SL_2(\mathbb{C})$ implies that the algebras $\mathcal{I}_{d_1,d_2}$, $\mathcal{C}_{d_1,d_2}$ are finitely generated $\mathbb{Z}$-graded algebras

$$\mathcal{I}_{d_1,d_2} = (\mathcal{I}_{d_1,d_2})_0 + (\mathcal{I}_{d_1,d_2})_1 + \cdots + (\mathcal{I}_{d_1,d_2})_i + \cdots ,$$

$$\mathcal{C}_{d_1,d_2} = (\mathcal{C}_{d_1,d_2})_0 + (\mathcal{C}_{d_1,d_2})_1 + \cdots + (\mathcal{C}_{d_1,d_2})_i + \cdots ,$$

where each of subspaces $(\mathcal{I}_{d_1,d_2})_i$, $(\mathcal{C}_{d_1,d_2})_i$ of simultaneous invariants and covariants of degree i is finite dimensional.

The formal power series $\mathcal{PI}_{d_1,d_2}(z), \mathcal{PC}_{d_1,d_2}(z) \in \mathbb{Z}[[z]]$,

$$\mathcal{PI}_{d_1,d_2}(z) = \sum_{i=0}^{\infty} \dim((\mathcal{I}_{d_1,d_2})_i) z^i, \mathcal{CI}_{d_1,d_2}(z) = \sum_{i=0}^{\infty} \dim((\mathcal{C}_{d_1,d_2})_i) z^i,$$

are called the Poincaré series of the algebras of simultaneous invariants and covariants. The finite generation of the algebras $\mathcal{I}_{d_1,d_2}$, and $\mathcal{C}_{d_1,d_2}$ implies that their Poincaré series are expansions of certain rational functions. Here we consider the problem of computing these rational functions efficiently.

The Poincaré series calculation was an important object of research in classical invariant theory of the 19th century. For the cases $d \leq 10$, $d = 12$ the Poincaré series of the algebra of invariants for the binary form of degree d were calculated by Sylvester and Franklin, see [1], [2]. Relatively recently, Springer [3] derived the explicit formula for computing the Poincaré series of the algebras of invariants of the binary d-forms. This formula has been used by Brouwer and Cohen [4] for $d \leq 17$ and also by Littelmann and Procesi [5] for even $d \leq 36$. In [6] the Poincaré series of algebras of simultaneous covariants of two and three binary forms of small degrees are calculated.

We have found Sylvester-Cayley type formulas for calculation of $\dim(\mathcal{I}_{d_1,d_2})_i$, $\dim(\mathcal{C}_{d_1,d_2})_i$ and Springer type formulas for calculation of $\mathcal{PI}_{d_1,d_2}(z)$ and $\mathcal{PC}_{d_1,d_2}(z)$. Namely,

$$\mathcal{PI}_{d_1,d_2}(t) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1-z^2}{(tz^{d_2-d_1}, z^2)_{d_1+1} (t, z^2)_{d_2+1}} \frac{dz}{z},$$

$$\mathcal{PC}_{d_1,d_2}(t) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1+z}{(tz^{d_2-d_1}, z^2)_{d_1+1} (t, z^2)_{d_2+1}} \frac{dz}{z}, d_2 \geq d_1.$$

By using the derived formulas, the Poincaré series for $d_1, d_2 \leq 20$ are computed.

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Characters of Lie algebra $\mathfrak{sl}_3$ and (q,p)-numbers

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The usual q-number $[n]_q := \dfrac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ can be interpreted as a specialized character of irreducible representation of the Lie algebra $\mathfrak{sl}_2$. The aim of this work is generalisation of above approach in the case of Lie algebra $\mathfrak{sl}_3$ to obtain relation of its representation characters with double (q,p)-deformed numbers $[x]_{q,p} := \dfrac{q^x - p^{-x}}{q - p^{-1}}$.

Let Γ_λ be the standard irreducible complex representation of $\mathfrak{sl}_3$ with the highest weight $\lambda = (m,n)$. Denote by Λ the weight lattice of all finite dimensional representations of $\mathfrak{sl}_3$, and let $\mathbb{Z}(\Lambda)$ be their group ring. The ring $\mathbb{Z}(\Lambda)$ is free $\mathbb{Z}$-module with the basis elements $e(\lambda)$, $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, $e(\lambda)e(\mu) = e(\lambda+\mu)$, $e(0) = 1$. Let Λ_λ be the set of all weights of the representation Γ_λ. Then the formal character $\mathrm{Char}(\Gamma_\lambda)$ is defined (see [1]) as formal sum $\sum_{\mu\in\Lambda_\lambda} n_\lambda(\mu)e(\mu) \in \mathbb{Z}(\Lambda)$, here $n_\lambda(\mu)$ is the multiplicities of the weight μ in the representation Γ_λ. By replacing $e(m,n) := q^n p^m$ we obtain the specialized expression for the character $\mathrm{Char}(\Gamma_{(n,m)})$. For example $[1,0]_{q,p} = q + p^{-1} + \frac{p}{q}$, $[0,1]_{q,p} = p + \frac{q}{p} + q^{-1}$. More generally, $[n,0]_{q,p} = H_n(q, p/q, 1/p)$, $[0,n]_{q,p} = H_n(p, q/p, 1/q)$, where $H_n(x,y,z)$ is the complete symmetrical function $H_n(x,y,z)$, (see [2] for details). The following statements hold:

Theorem 1.

$$(i)\qquad [n,m]_{q,p} = [n,0]_{q,p}[0,m]_{q,p} - [n-1,0]_{q,p}[0,m-1]_{q,p},$$

$$(ii)\qquad [n,0]_{q,p} - (pq^{-1})[n-1,0]_{q,p} = [n+1]_{q,p}, [0,0]_{q,p} = 1,$$

$$(iii)\qquad [n,0]_{q,p} = \sum_{k=1}^{n+1} (pq^{-1})^{n-k+1}[k]_{q,p},$$

$$(iv)\qquad [n-1,0]_{q,p} = \frac{[n+1]_{q,p} - (qp^{-1})^2[n]_{q,p} - (pq^{-1})^n[1]_{q,p}}{[2]_{q,p} - (pq^{-1})^2 - (pq^{-1})},$$

$$(iv)\qquad [n-1,0]_{q,q} = \frac{[n+1]_q[m+1]_q[m+n+2]_q}{[2]_q}.$$

Note that for $p = q = 1$ the (q,p)-number $[n,m]_{q,p}$ is equal to the dimension of the representation $\Gamma_{(m,n)}$.

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Алгебраические основы и приложения преобразований Грея

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Коды Грея, предложенные в середине XX века в ответ на запросы инженерной практики относительно построения оптимальных по критерию минимума ошибки неоднозначности преобразователей типа "угол-код" [1], на заре своего появления привлекли к себе внимание не только исследователей математиков, но и широкого круга разработчиков разнообразной аппаратуры. Отличительная особенность кодов Грея состоит в том, что в двоичном пространстве (или в двоичной системе счисления) при переходе от изображения одного числа к изображению соседнего старшего или соседнего младшего числа происходит изменение цифр (1 на 0, или наоборот) только в одном разряде числа.

За более чем пятидесятилетнюю историю своего развития теория кодов Грея претерпела незначительные изменения. По-видимому, оказались вне поля зрения, как математиков, так и разработчиков аппаратуры, возможности построения кодов, инверсных по направлению формирования классическим кодам Грея. В известной схеме [1] процесс формирования прямых и обратных кодов Грея развивается по направлению слева направо; при этом старший (левый) разряд преобразуемой кодовой комбинации сохраняется неизменным. Вместе с тем можно построить схему преобразования, в общем, m-ичных равномерных кодов, обратную по направлению классическому (левостороннему) преобразованию Грея [2]. В таком классе преобразований, который назван правосторонним, при прямом и обратном преобразованиях сохраняется неизменным значение младшего (правого) разряда преобразуемого числа.

Комбинация лево- и правостороннего преобразований Грея (как прямого, так и обратного), образующих совокупность простых кодов Грея, совместно с оператором инверсной перестановки (представляющим собой квадратную $(0, 1)$-матрицу, в которой элементы вспомогательной диагонали равны единице, а остальные — нулю, послужила основой построения комбинированных или составных кодов Грея. Применение как простых, так и составных кодов Грея оказалось весьма успешным в задачах определения структуры и взаимосвязи дискретных симметричных систем Виленкина-Крестенсона функций (ВКФ), частным случаем которых являются системы дискретных экспоненциальных функций и системы функций Уолша. И, тем не менее, не для всех порядков систем ВКФ удается "связать" с помощью упомянутых выше преобразований Грея полное множество симметричных систем ВКФ. Возникает так называемая проблема кластеризации, которая, для примера, проявляется в том, что только 128 из 448 симметричных систем функций Уолша 16-го порядка оказалось возможным синтезировать с помощью простых и составных кодов Грея. Обозначенную проблему кластеризации удалось разрешить введением так называемых обобщенных кодов Грея [3].

В докладе обсуждаются теоретические основы построения алгебраической структуры, состоящей из конечного числа элементов (индикаторных матриц систем ВКФ), на множестве которых определена бинарная операция типа умножения, удовлетворяющая основным аксиомам группы: операция ассоциативна; гарантирует единицу и обратные элементы; замкнута для каждой пары элементов, входящих в группу. Рассматриваются различные направления применения преобразований Грея: в криптографии, теории и практике дискретного спектрального анализа сигналов и изображений, структурном синтезе пересчетных схем [4] и др.

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On the average value of Dirichlet's convolutions of the functions $\tau(n)$ and $r(n)$ in arithmetic progressions

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Let $\tau(n)$ and $r(n)$ denote the number of divisors and the number of representations as a sum of two squares of natural n. For each of these functions we have the asymptotic formula of summatory functions in the arithmetic progressions modulo q which are nontrivial for $q \le x^{\frac{2}{3}-\varepsilon}$, where x is a length of sum.

The application of such asymptotic formulas allows to obtain in asymptotic formulas for the sums

$$\sum\nolimits^{*} r(m)r(n), \ \sum\nolimits^{*} r(m)\tau(n), \ \sum\nolimits^{*} \tau(m)\tau(n)$$

(here the $*$ indicates that the summation runs over all positive integers m, n for which $mn \equiv \ell(mod\ q)$ and $mn \ll x$)

the estimates of error terms $R(x) \ll x^{\frac{3}{5}+\varepsilon} q^{\frac{1}{5}}$, $\varepsilon > 0$.

Using the estimates of the normal Kloosterman sums over $\mathbb{Z}[i]$ and 3-dimension Kloosterman sums over $\mathbb{Z}$ we improve the appropriate asymptotic formulas:

$$S^{(1)}(x;\ell,q) = \sum\nolimits^{*} r(m)r(n) = \frac{x}{q} P_q^{(1)}\left(\log\frac{x}{q^2}\right) + O\left(x^{\frac{1}{2}+\varepsilon} q^{\frac{1}{4}}\right)$$

$$S^{(2)}(x;\ell,q) = \sum\nolimits^{*} r(m)\tau(n) = \frac{x}{q} P_q^{(2)}\left(\log\frac{x}{q^2}\right) + O\left(x^{\frac{1}{2}+\varepsilon} q^{\frac{1}{4}}\right)$$

$$S^{(1)}(x;\ell,q) = \sum\nolimits^{*} \tau(m)\tau(n) = \frac{x}{q} P_q^{(3)}\left(\log\frac{x}{q^2}\right) + O\left(x^{\frac{1}{2}+\varepsilon} q^{\frac{1}{4}}\right)$$

where $P_q^{(j)}(u)$ is a polynomial of degree j with the highest term $C_0^{(j)}$, $(\log q)^{-1} \ll C_0^j \ll \log q$, $j = 1, 2, 3$.

We also prove the following theorem

Theorem 1. *Let $x \in \mathbb{R}$, $q \in \mathbb{N}$, $x^{\frac{1}{4}} \leq q$. Then for almost all ℓ, $1 \leq \ell \leq q$, $(\ell, q) = 1$ we have*

$$|S^{(j)}(x;\ell,q) - \frac{x}{q}P_q^{(j)}\left(\log\frac{x}{q^2}\right)| \ll \begin{cases} \left(\frac{x}{q}\right)^{\frac{4}{5}} \cdot x^{\varepsilon} & if\ q > x^{\frac{1}{3}}; \\ x^{\frac{7}{10}+\varepsilon} q^{-\frac{3}{5}} & if\ x^{\frac{1}{4}} < q \leq x^{\frac{1}{3}}. \end{cases}$$

This result is an analog of Theorem 1 from [1].

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Totally conjugate-orthogonal quasigroups

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It is known that every quasigroup (Q, A) has six (not necessarily distinct) conjugates (or parastrophes): A, ${}^rA$, ${}^lA$, ${}^{rl}A$, ${}^{lr}A$, ${}^sA$ where ${}^rA(x,y) = z \Leftrightarrow A(x,z) = y$, ${}^lA(x,y) = z \Leftrightarrow A(z,y) = x$ and ${}^sA(x,y) = A(y,x)$ which are quasigroups.

Two quasigroups (Q, A) and (Q, B) are called orthogonal if the system $\{A(x,y) = a, B(x,y) = b\}$ has a unique solution for all $a, b \in Q$.

We say that a quasigroup (Q, A) is a *totally conjugate-orthogonal quasigroup* (briefly, a TotCO-quasigroup) if all its conjugates are pairwise orthogonal. In this case the system $\sum = \{A, {}^rA, {}^lA, {}^{rl}A, {}^{lr}A, {}^sA\}$ is an orthogonal set of quasigroups. Any quasigroup of $\sum$ is also a TotCO-quasigroup.

A quasigroup (Q, A) is called a T-quasigroup if there exist an abelian group $(Q, +)$, its automorphisms φ , ψ and an element $a \in Q$ such that $A(x,y) = \varphi x + \psi y + a$.

Necessary and sufficient conditions that a T-quasigroup be a TotCO-quasigroup are established.

In [1] it is proved that there exist infinite TotCO-quasigroups. For finite quasigroups we prove the following

Theorem 1. *There exist TotCO-quasigroups of any order n which is not divided by 2, 3, 5 and 7.*

Corollary 1. *There exists a TotCO-quasigroup of order $n = p^k$ where $p \neq 2,3,5,7$ is a prime number, $k \geq 1$.*

The computer search confirms that do not exist TotCO-quasigroups of order $n = 7$ among quasigroups of the form $A(x,y) = ax + by (mod n)$.

A conjugate-orthogonal quasigroup graph, in which the conjugates are the vertices and two vertices are joint if and only if the respective conjugates are orthogonal, corresponds to every quasigroup.

Corollary 2. *For any $n = p_1^{k_1} p_2^{k_2} ... p_s^{k_s}$ where p_i, $i = 1,2,...,s$, are prime numbers not equal to 2, 3, 5, 7, $k_i \geq 1$, there exists a quasigroup of order n realizing the complete graph K_6.*

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Postulates of Sushkevich and linear quasigroups

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In [1] A.K. Sushkevich proved that if in a quasigroup $(Q,\cdot)$ the identity (now known as the postulate (A) of Sushkevich [2]) $xy\cdot z = x(y\circ z)$ is carried out, then the quasigroup $(Q,\cdot)$ is isotopic to the group $(Q,\circ)$.

If a quasigroup $(Q,\cdot)$ satisfies the identity (known as the postulate (B) of Sushkevich [2]) $xy\cdot z = x(y\cdot\alpha z)$ where α is some substitution of the set Q, then the quasigroup $(Q,\cdot)$ is also isotopic to a group.

We consider the following more general identity in any quasigroup $(Q,\cdot)$:

$$\alpha_1(\alpha_2(x\otimes_1 y)\otimes_2 z) = \alpha_3 x\otimes_3 \alpha_4(\alpha_5 y\otimes_4 \alpha_6 z) \tag{1}$$

where α_i, $i = 1,2,...,6$, are some substitutions of the set Q, $(\otimes_k) = (\cdot)$ or $(\otimes_k) = (*)$, $x * y = y\cdot x$, $k = 1,2,...,4$.

Theorem. *If in a quasigroup $(Q,\cdot)$ the identity (1) holds for some substitutions α_i, $i = 1,2,...,6$, then the quasigroup $(Q,\cdot)$ is isotopic to a group. Converse, if a quasigroup $(Q,\cdot)$ is isotopic to a group, then in this quasigroup the identity (1) is carried out for some suitable substitutions α_i, $i = 1,2,...,6$.*

A quasigroup $(Q,\cdot)$ is called linear (alinear) over a group if there exist a group $(Q,\circ)$, its automorphisms (antiautomorphisms) φ,ψ and an element $a\in Q$ such that $x\cdot y = \varphi x\circ a\circ\psi y$.

It is established a number of identities with the substitutions in a quasigroup which are particular cases of (1) and guarantee linearity, alinearity, semilinearity, semialinearity or linearity of a mixed type of this quasigroup over a group or over an abelian group. These results permit to describe infinite number of identities which supply isotopy of a quasigroup to a group or its linearity of a given type.

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Analog of Cohen's theorem for distributive Bezout domains

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Let R be a right distributive Bezout domain. Remark, that domain is a left (right) distributive if its lattice of right (left) ideals is distributive [1]. The right ideal P of ring R is called prime right ideal if it follows from $aRb \subseteq P$, that $a \in P$ or $b \in P$. The right Bezout domain is a ring without zero divisors in which every finitely generated right ideal is principal.

Theorem 1. *Distributive right Bezout domain in which any prime right ideal is principal is a principal right ideals domain.*

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The speciality of the structure of two-sided ideals of elementary divisor domain

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Let ring R be an elementary divisor domain. Remind, that integer domain R is an elementary divisor domain, if any matrix A over R possesses a canonical diagonal reduction, that is, there exist invertible matrices R, P and Q of corresponding sizes, such that $PAQ = (d_{ij})$ is the diagonal matrix, such that $Rd_{i+1,i+1}R \subseteq d_{ii}R \cap Rd_{ii}$ [1]

Theorem 1. *In elementary divisor domain the intersection of all nontrivial two-sided ideals is equal to zero.*

Theorem 2. *There is no elementary divisor domain with finite number of nontrivial two-sided ideals.*

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Nagata's type automorphisms as the exponents of three root locally nilpotent derivations

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Locally nilpotent derivations from a Lie algebra sa_n of the special affine Cremona group are investigated in a connection with the root decomposition of sa_n relative to the maximal standard torus.

Earlier it was proved (see [1]) that all root locally nilpotent derivations are elementary ones. It immediately follows from this, that a sum of two roots is locally nilpotent if it is a sum of two elementary derivations and has the triangular form only. A class of three root locally nilpotent derivations is more interesting, because the well-known Nagata's and Anick's exotic automorphisms of polynomial algebra can be obtained as the exponents of such derivations. Below we use the bold font for the monomials of the form $\mathbf{x}_3^{\mathbf{k}_3} = x_3^{k_3} x_4^{k_4} \dots x_n^{k_n}$.

Theorem 1. *Three root locally nilpotent derivations of polynomial algebra* $\mathbb{F}[x_1, \dots, x_n]$ *over the field* $\mathbb{F}$ *of characteristic zero by renaming the variables accurate within a constant multiple can all be reduced to the derivation*

$$\left((k_2+1)\frac{\alpha}{\beta}x_1\mathbf{x}_3^{\mathbf{k}_3} + x_2^{k_2+1}\mathbf{x}_3^{\mathbf{u}_3}\right)\left(\beta x_2^{k_2}\frac{\partial}{\partial x_1} - \alpha\mathbf{x}_3^{\mathbf{k}_3-\mathbf{u}_3}\frac{\partial}{\partial x_2}\right), \tag{1}$$

where all coordinates of the vectors $2\mathbf{k}_3 - \mathbf{u}_3$ *are nonnegative integers, or to the triangular form.*

Using the I. Shestakov's and Umirbaev's results [2] we get a series of wild automorphisms which are the exponents of three root locally nilpotent derivations.

Theorem 2. *Let* D *be a derivation of the polynomial algebra in three variables of the form* (1) *and* $k_3 > u_3$*, then the automorphism of polynomial algebra* $\mathbb{F}[x_1, x_2, x_3]$

$$\exp(D) = \left\langle x_1 - \frac{\beta}{k_2+1}x_3^{-m}\left(\left(x_2 - \frac{1}{\beta}\Delta x_3^m\right)^{k_2+1} - x_2^{k_2+1}\right), x_2 - \Delta x_3^m\frac{1}{\beta}, x_3\right\rangle,$$

where $m = k_3 - u_3, l = u_3, \Delta = \left((k_2+1)x_1x_3^m + \beta x_2^{k_2+1}\right)x_3^l \in KerD$*, is wild if and only if* $m > 0, k_2 > 0$.

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Binomial properties with significant consequences about Fermat's equation and on the cosine rule

Enzo Bonacci$^\flat$, Mario De Paz*

There are some unexplored properties of the binomial expansion with relevant influences on Fermat's equation and Cosine Rule.

The talk consists of three steps:

1. Proving unexplored properties of Pascal's triangle;
2. Analyzing the consequences of some binomial properties in limiting Fermat's triple until an almost impossible condition of existence;
3. Using other binomial properties to extend the Law of Cosines to powers greater than two, i.e., on synclastic surfaces.

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Conjugation of finite-state automorphisms of regular rooted trees

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We consider the conjugacy problem in the group $Aut\ T$ of all automorphisms of a regular rooted tree T and in its subgroup $Aut_f T$ of all finite-state (automatic) automorphisms. The conjugacy classes of the group $Aut\ T$ are described in [1]. An example of two finite-state automorphisms, which are conjugated in $Aut\ T$ and not conjugated in $Aut_f\ T$, is presented in [2]. At the same time, it is shown in [3] that if the finite-state automorphisms have finite order and they are conjugated in $Aut\ T$ then they are conjugated in $Aut_f\ T$. The conjugacy problem in the group $Aut_f\ T$ is an open problem.

One important class of finite-state automorphisms is a class of bounded automorphisms defined by S. Sidki. The set of all bounded automorphisms forms a group called the group of bounded automata. Many interesting groups like the Grigorchuk group, Gupta-Sidki groups, Sushchansky groups, iterated monodromy groups of post-critically finite polynomials are subgroups of the group of bounded automata. We prove the following

Theorem 1. *Two bounded automorphisms are conjugated in the group $Aut\ T$ if and only if they are conjugated in the group $Aut_f\ T$, and this conjugacy problem is solvable.*

Theorem 2. *The conjugacy problem is solvable in the group of bounded automata.*

The solution of the conjugacy problem is connected to an interesting problem about nonnegative matrices, related to the joint spectral radius and subradius of a set of nonnegative matrices.

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Classification of NP-critical posets

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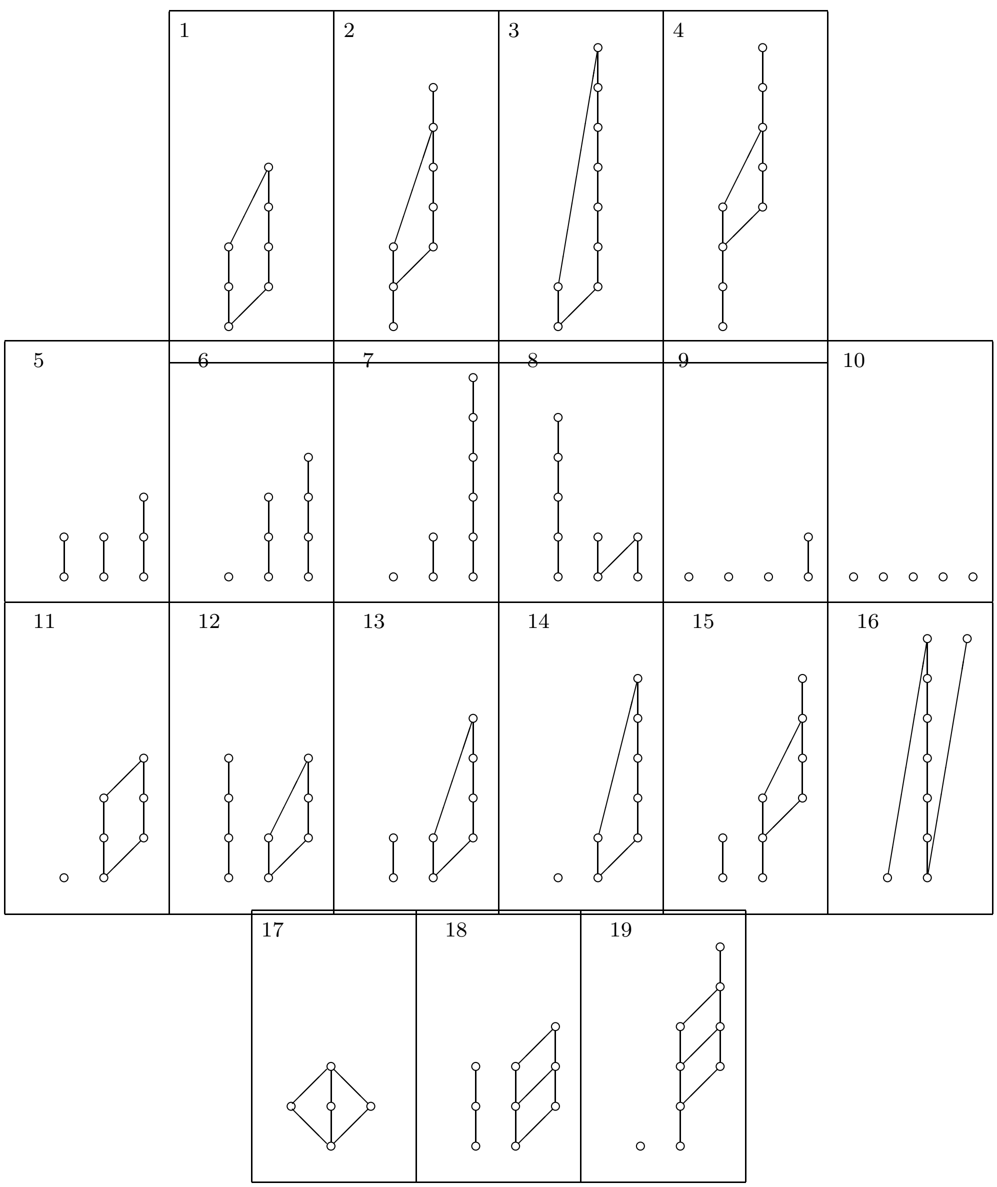
We describe the finite posets with non-negative Tits form, all proper subposets of which have yet negative Tits form (abbreviated: NP-critical posets). In particular we prove the next statement.

Theorem. *Any NP-critical poset is (min,max)-isomorphic to a self-dual one, and every self-dual NP-critical poset is isomorphic to one from the following list:*



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Quadratic forms of semigroups generated by idempotents with partial null multiplication

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Let $I = \{1, 2, \ldots, n\}$ and let J be a subset of $I \times I$ without diagonal elements (i, i). We define $S(I, J)$ to be the semigroup with generators e_i, where $i \in I \cup 0$, and the following relations:

1) $e_0 = 0$ ($e_0 e_i = e_i e_0 = 0$ for any $i \in I$);

2) $e_i^2 = e_i$ for any $i \in I$;

3) $e_i e_j = 0$ for any pair $(i, j) \in J$.

The set of all semigroups of the form $S(I, J)$ is denoted by $\mathcal{I}$. We call $S(I, J) \in \mathcal{I}$ *a semigroup generated by idempotents with partial null multiplication.*

Let $S = S(I, J) \in \mathcal{I}$ and $\overline{J} = \{(i, j) \in (I \times I) \setminus J \,|\, i \neq j\}$. With the semigroup $S = S(I, J)$ we associate the quadratic form $f_S(z) : \mathbb{Z}^m \to \mathbb{Z}$ in the following way:

$$f_S(z) = \sum_{i \in I} z_i^2 - \sum_{(i,j) \in \overline{J}} z_i z_j.$$

We call $f_S(z)$ *the quadratic form of the semigroup* S.

We prove the following theorems.

Theorem 1. *A semigroup $S(I, J)$ is of finite representation type over k if and only if its quadratic form is positive (then $S(I, J)$ is finite).*

Theorem 2. *Let $S(I, J)$ be a finite semigroup. Then $S(I, J)$ is tame over k if its quadratic form is nonnegative, and wild, otherwise.*

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On triangular similarity of nilpotent matrices

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One studies the problem of reducing triangular matrices (over a field) by triangular similarity. In particular, one proves that any triangular selfannihilating matrix is triangular similar to a monomial matrix, and gives an explicit practicable algorithm of such reducing.

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Биномиальные системы счисления

А. А. Борисенко

Предлагаются биномиальные системы счисления. Характерным их свойством является способность генерирования ими комбинаторных объектов, основанных на сочетаниях, сочетаниях с повторениями, композициях и других подобных объектах, в частности, помехоустойчивых кодов.

Теория этих систем счисления подробно рассмотрена в работах [1]-[3]. Их важной практической особенностью является способность перебирать комбинаторные объекты, решая тем самым задачу комбинаторного перебора, которая особенно часто используется в задачах комбинаторной оптимизации.

Относительно недавно были разработаны значительно более сложные, чем линейные биномиальные системы счисления, — *матричные* системы счисления, которые способны генерировать комбинаторные объекты с большим быстродействием и надежностью, чем обычные линейные системы [2]. Однако они представляют, на взгляд автора, и интересный математический объект, исследование которого пока что еще далеко от своего завершения.

Матричные биномиальные системы счисления представляют собой двоичные матрицы, содержащие k столбцов и $(n-k)$ строк, где n и k — параметры биномиальных коэффициентов C_n^k, которые определяют диапазон данной системы счисления. В этой матрице $x_{ij} \notin \{0,1\}$ представляют цифры матричной биномиальной системы счисления, биномиальные коэффициенты C_{i+j}^j которой образуют весовые значения этих цифр.

Числовая функция в этом случае для матричной биномиальной системы счисления будет иметь следующий вид:

$$F = \sum_{i=0}^{n-k} \sum_{j=1}^{k} x_{ij} C_{i+j}^j$$

С ее помощью биномиальная матрица преобразуется в соответствующий номер и обратно номер преобразуется в биномиальную матрицу.

Так как x_{ij} может принимать лишь два значения 0 и 1, то при записи конкретных чисел в матричной биномиальной форме их можно представить, записывая лишь те элементы матрицы, где $x_{ij} = 1$, как, например, показано в таблице 2, получив при этом матрицу весовых коэффициентов — *весовую* матрицу.

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Automorphisms of Chevalley groups over rings

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Let $G_\pi(\Phi, R)$ be a Chevalley group of type Φ over a commutative ring R with a unit, $E_\pi(\Phi, R)$ be an elementary subgroup in $G_\pi(\Phi, R)$. We describe automorphisms of groups $E_\pi(\Phi, R)$ and $G_\pi(\Phi, R)$ over local commutative rings with 1/2, for root systems A_l, D_l, E_l, $l > 1$. Similar results for Chevalley groups over fields were proved by R. Steinberg for finite fields and J. Hamphris for infinite fields. Description of automorphisms of Chevalley groups over different commutative rings were studied by various authors, explicitly by Borel–Titz, Carter–Chen Yu, Chen Yu, E. Abe, Anton A. Klyachko.

Let us define four types of automorphisms of a Chevalley group $G_\pi(\Phi, R)$.

1. Let $C_G(R)$ be a center of the group $G_\pi(\Phi, R)$, $\tau : G_\pi(\Phi, R) \to C_G(R)$ a homomorphism of groups. Then the mapping $x \mapsto \tau(x)x$ from $G_\pi(\Phi, R)$ onto itself is called a *central automorphism* of the group $G_\pi(\Phi, R)$.

2. Let $\rho : R \to R$ be an automorphism of the ring R. The mapping $x \mapsto \rho(x)$ from $G_\pi(\Phi, R)$ onto itself is called a *ring automorphism* of the group $G_\pi(\Phi, R)$.

3. Let S be a ring containing R, g be an element of $G_\pi(\Phi, S)$, normalizing $G_\pi(\Phi, R)$. Then the mapping $x \mapsto gxg^{-1}$ is denoted φ_g and is called an *inner automorphism, induced by an element g*.

4. Let δ be an automorphism of the root system Φ such that $\delta\Delta = \Delta$. Then there exists a unique automorphism of the group $G_\pi(\Phi, R)$ such that for any $\alpha \in \Phi$ and $t \in R$ an element $x_\alpha(t)$ is mapped to $x_{\delta(\alpha)}(\varepsilon(\alpha)t)$, where $\varepsilon(\alpha) = \pm 1$ for all $\alpha \in \Phi$ and $\varepsilon(\alpha) = 1$ for all $\alpha \in \Delta$ (*graph automorphism*).

Similarly we can define four types of automorphisms of the elementary subgroup $E(R)$.

An automorphism σ of the group $G_\pi(\Phi, R)$ (or $E_\pi(\Phi, R)$) is called *standard*, if it is a composition of introduced four types of automorphisms.

Theorem 1. *Let $G = G_\pi(\Phi, R)$ $(E_\pi(\Phi, R))$ be a (elementary) Chevalley group with a root system A_l, D_l, or E_l, $l \geqslant 2$, R a commutative local ring with 1/2. Then every automorphism of the group G is standard.*

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Some properties of the class $\mathcal{T}$ in the category of modules over different rings

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Definition. *A preradical functor (or simply a preradical) on $\mathcal{C}$ is a subfunctor of the identity functor on $\mathcal{C}$.*

As usual, all rings are associative with $1 \neq 0$, all modules are left and unitary. The category of left modules is denoted by $R - Mod$. The investigation is based on the monographs [1-3] and the papers [4,5].

The objects of the category $\mathcal{G}$ are the pairs (R, M), where R is a ring , $R \in R - Mod$. The morphisms of the category $\mathcal{G}$ are semilinear transformations $(\varphi, \psi)\colon (R_1, M_1) \to (R_2, M_2)$, where $\varphi\colon R_1 \to R_2$ is a surjective homomorphism and $\psi\colon M_1 \to M_2$ is a homomorphism of abelian groups, and $\forall m \in M_1, \forall r \in R_1 \quad \psi(rm) = \varphi(r)\psi(m)$.

Let T be an idempotent preradical functor on the category $\mathcal{G}$. Consider the class

$$\mathcal{T}(T) = \{(R, M) \mid T(R, M) = (R, M)\},\ where (R, M) \in Ob(\mathcal{G}).$$

Proposition 1. *The class $\mathcal{T}$ is closed under epimorphic images.*

Proposition 2. *The class $\mathcal{T}$ possesses the following property: if $(R, M_1) \in \mathcal{T}(T)$ and $(R, M_2) \in \mathcal{T}(T)$ then $(R, M_1 \oplus M_2) \in \mathcal{T}(T)$.*

Proposition 3. *Let $\mathcal{S}$ be a class of objects of the category $\mathcal{G}$, which is closed under epimorphic images and under direct sums (if they exist). Put*

$$T(R, M) = \sum \{(R, M_i) | (R, M_i) \subseteq (R, M), (R, M_i) \in Ob(\mathcal{S})\}.$$

Then T is an idempotent preradical.

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Free Category Generated by a Partial Semigroup

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It is well-known that for arbitrary category its set of morphisms forms a partial semigroup. Actually one of approaches of characterizing category is by its hom-sets[1]. Here we consider the opposite problem how to build category from arbitrary partial semigroup.

Given a partial semigroup S. There are a set of elements and a set of equations defining the composition law. We are going to build a category C_S from this partial semigroup. This means we should somehow extract the set of objects and morphisms. Obviously settings imply that carrier of S would be embedded into set of morphisms of category and category should have some universal properties of uniqueness.

The calculation of object set is as follows. It is purely graph-theoretic. We start with disjoint union $S + S$ as nodes set and set S as edges. Graph structure is defined by injections $i_{1,2} \colon S \to S + S$. For every equation $uv = w$ we should equate pairs of nodes $(i_2(u), i_1(v))$, $(i_1(u), i_1(w))$, $(i_2(v), i_2(w))$. We would end with graph G' whose nodes set would be the object set of our category. This construction also can be formulated categorically as coequalizer of three pairs of morphisms.

Using graph G' we can build term algebra T_S with respect to constraints of categorical composition and obtained object set. This would be a free category generated by graph G' [1]. Equations of the semigroup S generate a congruence relation $\cong$ on T_S. The quotient $T_S/_{\cong}$ is the set of morphism of the category being sought.

Proposition 1. *The defined construction is a free construction for erasing functor from category of small categories to category of partial semigroups.*

Proposition 2. *If the set of equations of semigroup S is finite then congruence relation $\cong$ is semidecidable.*

The construction is quite straight-forward. The most complex and important aspect of it is decidability of equality, for categorical properties are based on equations. Problem with decidability is that terms don't have normal form, which is a common case in mathematics[2].

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The extensions of proper locally finite subgroups with quasicentral sylow 2-subgroups by the quaternion group

I. Chernenko

The subgroup A of the group G is called quasicentral in G, if all subgroups of the subgroup A are invariant in the group G.

The groups being extensions of the quasicircle subgroup by the quaternion group are investigated in the work [1]. The finite 2-groups, which are extensions of the quasicentral subgroup by the quaternion group of order 8 are described in the work [2].

In our report the theorem 1 is devoted to complete description of the groups G being extensions of proper locally finite subgroups D with quasicentral in G sylow 2-subgroups by the quaternion group of order 8.

Theorem 1. *All groups G being extensions of the proper locally finite subgroups D with quasicentral in G Sylow 2-subgroups by the quaternion group of order 8 are*

$$G = N\lambda(C \cdot H), \tag{1}$$

where N - locally finite holl normal subgroup in G without involution, C - quasicentral in G 2-subgroup, $H = K\cdot < b >$ - finite 2-generated 2-subgroup, $K =< c >< a >$, $\mid a \mid = 2^{\alpha},\ \alpha > 1,\ \mid b \mid = 2^{\beta},\ \beta > 1,\ [\ a, b\ ] = c,\ < c >= H',\ \mid c \mid = 2^{\delta},\ \delta > 0,\ a^{-1}ca = c,\ D = N \times C,\ C \cap H = D \cap H = Z,\ a^2b^2 = f \in Z < \Phi(H),$ H - group of one of the theorem 2 types of the work [2].

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On groups whose subgroups are either subnormal or pronormal

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In some sense, many important types of subgroups have their antipodes, i. e. subgroups that have completely opposite properties. For normal subgroups, such antipodes are contranormal subgroups and especially abnormal subgroups. Recall, that a subgroup H of a group G is contranormal if $H^G = G$. A subgroup H of a group G is called abnormal if for any element g of G the subgroup $< H, H^g >$ contains g. Pronormality is a generalization of both abnormality and normality. A subgroup H of a group G is called pronormal if for every element g of G the subgroups H and H^g are conjugate in $< H, H^g >$. Of course, there is no such contrast between subnormal and pronormal subgroups as between contranormal and subnormal subgroups is. However, since a pronormal subgroup is subnormal if and only if it is normal, we can consider pronormality and subnormality as some kind of antipodes.

In the paper [1], locally graded periodic and locally soluble non-periodic groups with all pronormal subgroups were completely described. The groups whose all non-normal subgroups are abnormal were investigated in [2] and [3]. P. Legovini in [4, 5] studied finite groups whose all subgroups either subnormal or pronormal. The example constructed by A.Yu.Olshanskii [6, chapter 9]; shows that there is no hope to describe all infinite such groups. In order to get some constructive results one need to choose carefully an appropriate class of infinite groups in which obtaining meaningful results would be possible. The class of locally (soluble - by - finite) groups is very useful for this purpose. We proved the following

Theorem 1. *Let G be a locally (soluble - by - finite) group whose all subgroups are either subnormal or pronormal. Then G contains a normal periodic subgroup T such that G/T is nilpotent and torsion free.*

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About distributive property of lattices of semirings of continuous functions

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Lattices of congruences of semirings of continuous real-valued functions on topological space are researched. Let $C^+(X)$ be a semiring of all continuous non-negative functions defined on any topological space X with standard addition and multiplication operations. If we take the operation of maximum $\vee$, instead of addition, we will get idempotent semiring $C^\vee(X)$ and idempotent semifield $U^\vee(X)$. The set $\operatorname{Con} S$ of all congruences of semiring S is a lattice.

In 1998 in article [1] it is proved, that if the lattice $\operatorname{Con} C^+(X)$ or the lattice $\operatorname{Con} U(X)$ is distributive, then X is F-space. The natural question is the validity of the inverse implication. This question for the lattice $\operatorname{Con} U(X)$ is solved positively in 2003 by D. V. Shirokov [2, theorem].

We have solved this problem for the lattice $\operatorname{Con} C^+(X)$.

Theorem. *Lattices of congruences of semirings of continuous functions $C^+(X)$ and $C^\vee(X)$ on F-space X have a distributive property.*

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Derivation Mappings on non-commutative rings

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We define on an arbitrary ring A a family of mappings $\sigma = (\sigma_{x,y})$. These mappings are subscripted by elements of multiplicative monoid G. The assigned properties allow to call these mappings derivations of the ring A. The operations of differentiation defined traditionally on a ring [1], [2], [3] are particular examples in our case.

We construct a monoid algebra $A < G >$ by means of the family σ. In [5] we construct a category $\mathcal{C}$ in which $A < G >$ as an object of it satisfies the universal property. Therefore, it could be applied the well-known construction [1] of skew polynomials of one or several variables over non-commutative ring A.

We describe completely the derivation mappings in the case of a monoid generated by two elements [6]. This case is important especially for the theory of skew polynomials in one variable. The obtained results concerning this special case extend and generalize some related results of T. H. M. Smits in [4] (see also [2]).

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Modules over group rings of locally soluble groups with rank restrictions on some systems of subgroups

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The investigation of modules over group rings is an important direction in modern Algebra. Many interesting results were obtained in this direction. Noetherian modules over group rings is a broad class of modules over group rings. Remind that a module is called Noetherian if partially ordered set of all submodules of this module satisfies the maximality condition. It should be noted that many problems of Algebra require the investigation of some specific Noetherian modules. Naturally arose the question on investigation of modules over group rings which are not Artinian but which are similar to Noetherian in some sense.

Let A be a $\mathbf{R}G$-module where $\mathbf{R}$ is a commutative Noetherian ring. If $H \leq G$ then the quotient module $A/C_A(H)$ is called a cocentralizer of H in module A. The subject of the investigation of this paper is a $\mathbf{R}G$-module A where $\mathbf{R}$ is a commutative Noetherian ring, G is a locally soluble group of infinite rank (for different ranks), $C_G(A) = 1$, A is not Noetherian $\mathbf{R}$-module and for every proper subgroup H of infinite rank the cocentralizer of H in A is a Noetherian $\mathbf{R}$-module. A section p-rank for prime p and 0-rang for $p = 0$ are denoted by $r_p(G)$.

The main results are the following theorems.

Theorem 1. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group, $r_p(G)$ be infinite for some $p \geq 0$. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a Noetherian $\mathbf{R}$-module. Then G is a soluble group.*

Theorem 2. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group of infinite abelian section rank. Suppose that the cocentralizer of every proper subgroup of infinite abelian section rank in A is a Noetherian $\mathbf{R}$-module. Then G is a soluble group.*

Theorem 3. *Let A be a $\mathbf{R}G$-module, G be a locally soluble group of infinite special rank. Suppose that the cocentralizer of every proper subgroup of infinite special rank in A is a Noetherian $\mathbf{R}$-module. Then G is a soluble group.*

Theorem 4. *Let A be a $\mathbf{R}G$-module, G be a soluble group and $r_p(G)$ be infinite for some $p \geq 0$. Suppose that for every proper subgroup M such that $r_p(M)$ is infinite the cocentralizer of M in A is a Noetherian $\mathbf{R}$-module. Then G has the series of normal subgroups $H \leq N \leq G$ such that H is an abelian group, N/H is nilpotent and the quotient group G/N is isomorphic to C_{q^∞}, for some prime q.*

Theorem 5. *Let A be a $\mathbf{R}G$-module, G be a soluble group of infinite abelian section rank. Suppose that a cocentralizer of every proper subgroup of infinite abelian section rank in A is a Noetherian $\mathbf{R}$-module. Then G has the series of normal subgroups $H \leq N \leq G$ such that H is an abelian group, N/H is nilpotent and the quotient group G/N is isomorphic to C_{q^∞}, for some prime q.*

Theorem 6. *Let A be a $\mathbf{R}G$-module, G be a soluble group of infinite special rank. Suppose that a cocentralizer of every proper subgroup of infinite special rank in A is a Noetherian $\mathbf{R}$-module. Then G has the series of normal subgroups $H \leq N \leq G$ such that H is an abelian group, N/H is nilpotent and the quotient group G/N is isomorphic to C_{q^∞}, for some prime q.*

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On group of automorphisms of some finite inverse semigroup

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Let $N = \{1, 2, \ldots, n\}$. Denote by S_n and I_n respectively the symmetric group and the symmetric inverse semigroup on N. Let G be an arbitrary subgroup of the group S_n. Denote by $I(G)$ the set $\{\varphi \in I_n \,|\, \varphi \subseteq \psi \, for\, some\, \psi \in S_n\}$. It is easy to verify that $I(G)$ is an inverse subsemigroup of a semigroup I_n.

Theorem 1. *The group of automorphisms of an inverse semigroup $I(G)$ is isomorphic to a normalizer of a group G in the symmetric group S_n, that is, $Aut(I(G)) \cong N(G)$ (here $N(G)$ is a normalizer of G in S_n).*

Corollary 1. *Let G be a maximal subgroup of S_n and $G \neq A_n$, where A_n is the alternating group and $n \geq 5$. Then $Aut(I(G)) \cong G$.*

Example 1. *Let G be a subgroup of S_4 and $|G| = 8$. Then $Aut(I(G)) \cong G$.*

Example 2. *Let $G = S_n$, then $I(G) = I_n$. Since $N(S_n) = S_n$, then $Aut(I_n) \cong S_n$.*

Example 3. *Let $G = \Delta$, that is, G is a trivial group. Obviously, $N(\Delta) = S_n$ and $I(\Delta) \cong \mathfrak{B}(N)$, where $\mathfrak{B}(N)$ is Boolean of a set N. Then $Aut(\mathfrak{B}(N)) \cong S_n$.*

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On prolongations of quasigroups with middle translations

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Let $Q = \{1, 2, 3, \ldots, n\}$ be a finite set and $Q(\cdot)$ be a quasigroup on it. Permutations ϕ_i $(i \in Q)$ of Q such that $x \cdot \phi_i(x) = i$ for all $x \in Q$, are called tracks of the element i in a quasigroup $Q(\cdot)$.

Any mapping σ of a quasigroup $Q(\cdot)$ defines a new mapping $\bar{\sigma}$ on $Q(\cdot)$, called conjugated to σ, such that

$$\bar{\sigma}(x) = x \cdot \sigma(x)$$

If $\bar{\sigma}(Q) = Q$ then we say that σ is complete. A quasigroup having at least one complete mapping is called admissible.

Let $Q' = Q \cup \{(n+1)\}$ and ϕ be a permutations on the set Q then ϕ' is defined in such way:

$$\phi'(x) = \begin{cases} \phi(x) & \text{if } x \in Q \\ (n+1) & \text{if } x = (n+1) \end{cases}$$

In the construction for a prolongation of an admissible quasigroup $Q(\cdot)$ proposed by V. D. Belousov [2] the complete mapping σ of $Q(\cdot)$ and its conjugated mapping $\bar{\sigma}$ are used. The operation $(\circ)$ on Q' is defined by the formula:

$$x \circ y = \begin{cases} x \cdot y & \text{for } x, y \in Q,\ y \neq \sigma(x) \\ \bar{\sigma}(x) & \text{for } x \in Q,\ y = n+1 \\ \bar{\sigma}\sigma^{-1}(y) & \text{for } x = n+1,\ y \in Q \\ (n+1) & \text{for } x \in Q,\ y = \sigma(x) \\ (n+1) & \text{for } x = y = n+1 \end{cases} \tag{1}$$

We propose another form for this formula. For this purpose we introduce a new notion: **increment** $\Delta\sigma = \{\delta_1, \delta_2, \delta_3, \ldots, \delta_n\}$ to the quasigroup $Q(\cdot)$ with a complete permutation σ where $\delta_i = (q, \bar{\sigma}^{-1}(i))$ are transpositions.

Theorem. *Let $\{\phi_1, \ldots, \phi_n\}$ be tracks of $Q(\cdot)$ with a complete mapping σ and $\{\psi_1, \ldots, \psi_n, \psi_{n+1}\}$ be tracks of quasigroup $Q'(\circ)$ where $Q'(\circ)$ obtained from $Q(\cdot)$ by formula* (1). *Then*

$$\psi_i = \phi'_i \delta_i \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad \psi_{n+1} = \sigma'.$$

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On connection between quasi-uniform convergence by Arcela and Alexandrov

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The continuous functions defined over the topological group $(G, \Im)$ and having rang in the Frechet space Y (full metric locally convex topological space) are considered. Definition of Arcela [1] is formulated for a case of such functions.

Definition 1. *(by Arcela) The sequence of continuous functions* $\{f_n(t)\}_{n=1}^{\infty}$, $f_n(t): G \to Y$ *is called quasi-uniform converging by Arcela to function* $f(t)$, *if*

1. $\lim_{n\to\infty} f_n(t) = f(t), \forall t \in G$
2. $\forall \varepsilon > 0\ \forall N > 0\ \exists$ finite number of indexes $n_1, n_2, ...n_k$ such, that $\inf_{1\le i\le k} \rho(f(t), f_n(t)) < \varepsilon, t \in G, N < n_1 < n_2 < ... < n_k$.

Definition 2. *(by Alexandrov [2]) The pointwise convergence of sequence of maps* f_n *to the map from the topological space* G *into the metric space* Y *is called quasi-uniform convergence by Alexandrov, if for any natural number there is at most countable open covering* $\{\Gamma_1, \Gamma_2, ..., \Gamma_s, ...\}$ *of space* G *and such sequence* $n_1, n_2, ..., n_s, ...$ *of natural numbers, greater N, that*

$$\rho(f(x), f_{n_k}(x)) < \varepsilon, \forall x \in \Gamma_k.$$

The criterion of concurrence of these two convergence is formulated.

Theorem. *For continuous functions over the normal set* K *into the Frechet space* Y, *the quasi-uniform convergence by Alexandrov is equivalent to the quasi-uniform convergence by Arcela if and only if* K *is compact set in* $(G, \Im)$.

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Miniversal deformations of pairs of skew-symmetric forms

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We give a miniversal deformation of each pair of skew-symmetric matrices (A, B) under congruence; that is, a normal form with minimal number of independent parameters to which all matrices $(A + E, B + E')$ close to (A, B) can be reduced by transformations

$$(A + E, B + E') \mapsto \mathcal{S}(E, E')^T (A + E, B + E')\mathcal{S}(E, E'), \qquad \mathcal{S}(0, 0) = I, \tag{1}$$

in which $\mathcal{S}(E, E')$ smoothly depends on the entries of E and E'.

We formulate here our result only for pairs of 3-by-3 matrices because of the limitation on the size of the theses.

Theorem 1. *Let* (A, B) *be any pair of* 3×3 *skew-symmetric matrices. Then all matrices* $(A, B) + (E, E')$ *that are sufficiently close to* (A, B) *are simultaneously reduced by transformations* (1) *to one of* *the following forms*

$$
\begin{aligned}
&\text{(i)} \quad \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)+\left(\begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix},\begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}\right),\\
&\text{(ii)} \quad \left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&\lambda&0\\-\lambda&0&0\\0&0&0\end{bmatrix}\right)+\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\*&0&*\\*&*&0\end{bmatrix}\right), \quad \lambda\in\mathbb{C},\\
&\text{(iii)} \quad \left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}\right)+\left(\begin{bmatrix}0&*&*\\*&0&*\\*&*&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right),\\
&\text{(iv)} \quad \left(\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}\right)+\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right).
\end{aligned}
$$

Each of the pairs (i) $-$ (iv) *has the form* $(A_{\rm can}, B_{\rm can}) + \mathcal{D}$*, in which* $(A_{\rm can}, B_{\rm can})$ *is the canonical form of* (A, B) *for congruence* (*see* [1]) *and the stars in* $\mathcal{D}$ *are complex numbers that tend to zero as* (E, E') *tends to* $(0, 0)$*. The number of stars is smallest that can be attained by using transformations* (1)*; it is equal to the codimension of the congruence class of* (A, B)*.*

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Reduction of matrices over Kazimirsky domain

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It is known that simple Bezout domain is elementary divisor ring if and only if it is 2-simple [1]. Among 2-simple Bezout domain with stable range 1 we select new class of domains, namely Kazimirsky domain. Domain R we call Kazimirsky domain if for any non-zero $a, b \in R$ there exist such elements $x, y \in R$ and $u \in R$ that $ax + uby = 1$ [1].

Theorem 1. *2-simple domain with stable range 1 is Kazimirsky domain.*

Theorem 2. *Simple domain Bezout with stable range 1 is elementary divisor ring if and only if it's Kazimirsky domain.*

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Free subsemigroups in topological semigroups

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A subset of a topological space is said to be nowhere dense if the interior of its closure is empty. A subset is called meagre if it is a countable union of nowhere dense subsets. The complement of a meagre set is called co-meagre. The well-known Baire Category theorem says that a complete metric space is not a meagre set.

Let S be a topological semigroup. We will consider all products of topological semigroups as topological spaces with respect to product (Tykhonov) topology.

The semigroup S is said to be almost free if for each $n \geq 2$ the set

$$\{(s_1, \ldots, s_n) \in S^n | \{s_1, \ldots, s_n\} \text{ freely generates a free subsemigroup of } S\}$$

is not meagre and is co-meagre in S^n.

Theorem 1. *Let S be a complete metrizable topological semigroup with the identity e. Suppose that*
1) S contains a free dense subsemigroup;
2) Any neighbourhood of the identity is noncommutative.
Then S is almost free.

Let $\mathbb{N}$ be the set of all positive integers. Define $T(\mathbb{N})$ to be the semigroup of all transformations of $\mathbb{N}$ under usual composition.

Corollary 1. *The semigroup $T(\mathbb{N})$ is almost free.*

Let X be a finite alphabet, $|X| > 1$. Let $AS(X)$ be the semigroup of all automaton transformations over alphabet X (for definitions of automaton transformations see, for example [1]).

Corollary 2 [2]. *The semigroup $AS(X)$ is almost free.*

Let $T_\infty(q)$ denote the semigroup (under multiplication) of all finite dimensional, indexed by positive integers, upper triangular matrices over the finite field of order q.

Corollary 3 [3]. *The semigroup $T_\infty(q)$ is almost free.*

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Central series in the unitriangle automorphism group of the ring of two polynomials variable over the field of zero characteristic

Zh. Dovghey

During the latest time many publications (see for example [1]-[2]) are devoted to studying of the structure and properties of automorphism groups of the ring of two variable polynomials over a field.

Every automorphism of the polynomial ring $K[x, y]$ of two variables over the field K can be defined by the correspondence

$$x \to a_1(x, y), y \to a_2(x, y), \tag{1}$$

where $a_1(x, y)$ and $a_2(x, y)$ are such polynomials over K that for the mapping $K[x, y]$ in itself which is defined by (1) there exists the inverse (see for example [1]). An automorphism $< a_1(x, y), a_2(x, y) >$ is called unitriangle if

$$a_1(x, y) = x_1 + b_1, a_2(x, y) = x_2 + b_2(x_1), b_2(x) \in K[x_1].$$

All unitriangle automorphisms form the subgroup in group $AutK[x, y]$ which will be denoted by $UJ_2(K)$. In the talk we characterize upper and lower central series of the group $UJ_2(K)$ over the field of zero characteristic.

Theorem 1. *The commutator of the group $UJ_2(K)$ consists of automorphisms of the form*

$$< x_1, x_2 + b_2(x_1) >, b_2(x) \in K[x_1].$$

Arbitrary element of the commutator subgroup is a commutator of certain elements of the group $UJ_2(K)$. The central series of the group $UJ_2(K)$ stabilizes on the commutator subgroup.

Theorem 2. *k-th term of the upper central series of the group $UJ_2(K)$ $(k = 1, 2, ...)$ is a subgroup of all automorphisms of the form $< x_1, x_2 + b_2(x_1) >$, where $b_2(x_1) \in K[x_1]$, $deg(b_2(x_1)) \leq k - 1$. The upper central series of $UJ_2(K)$ has the length $\omega + 1$.*

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Titling theory and representations of linear groups over piecewise hereditary algebras

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A *linear group over an algebra A* is, by definition, the group of automorphisms $\mathrm{GL}(P, A)$ of a finitely generated projective A-module P. We use the technique of bimodule categories and tiliting theory to study the *dual space* $\widehat{G}$, i.e. the space of unitary representations of the group $G = \mathrm{GL}(P, A)$ in case, when A is a piecewise hereditary algebra over the field of complex numbers. Namely, we consider three cases:

1. A is *Dynkinian*, i.e. derived equivalent to the path algebra of a Dynkin quiver.
2. A is *Euclidean*, i.e. derived equivalent to the path algebra of a Euclidean (or, the same, extended Dynkin) quiver.
3. A is *tubular*, i.e. derived equivalent to the weighted projective line [3].

Theorem 1. *In the above mentioned cases the dual space $\widehat{G}$ contains an open dense subset $\widetilde{G}$ such that*

$$\widetilde{G} \simeq \prod_{i=1}^{r} \widehat{\mathrm{GL}}(d_i, \mathbb{C}) \times \prod_{j=1}^{s} \big(Q(m_j, U_j) \times (\mathbb{C}^*)^{m_j}\big),$$

where U_j are subsets of $\mathbb{C}$ with finite complements and $Q(m, U)$ denotes the factor of the set of vectors $\big\{(a_1, a_2, \dots, a_m) \,|\, a_i \in U,\ a_i \neq a_j \text{ if } i \neq j\big\}$ under the action of the symmetric group Σ_m.

Moreover,

1. *$s = 0$ if A is Dynkinian* [2].
2. *$s \leq 1$ if A is Euclidean* [1].

We conjecture that $s \leq 2$ if A is tubular.

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Finite p-groups ($p \neq 2$) with non-Abelian norm of Abelian non-cyclic subgroups

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Following [1], the norm N_G^A of Abelian non-cyclic subgroups is the intersection of normalizers of all Abelian non-cyclic subgroups of group G (on condition that the system of such subgroups is not empty). The norm N_G^A of Abelian non-cyclic subgroups is non-trivial, if it is non-Dedekind. The author studies finite p-groups ($p \neq 2$) with non-Abelian norm N_G^A of Abelian non-cyclic subgroup. Non-Abelian p-groups, in which all Abelian non-cyclic subgroups are invariant, were studied in [2] and were called $\overline{HA_p}$-groups.

Following [3], the condition of invariance of all non-cyclic subgroups is equivalent to the condition of invariance only of all Abelian non-cyclic subgroups in every locally finite p-group ($p \neq 2$). Non-Abelian p-groups, in which all non-cyclic subgroups are invariant, were studied in [3] and were called $\overline{H_p}$-groups.

Locally finite p-groups ($p \neq 2$) with non-Abelian norm N_G of non-cyclic subgroups were studied in [4]. Class of non-cyclic groups is wider than the class of Abelian non-cyclic groups, that is why $N_G \subseteq N_G^A$. Taking into account results of [4], we obtain, that $N_G^A = N_G$ in locally finite p-group ($p \neq 2$) G, when the non-cyclic norm N_G is non-Abelian.

If $G = N_G^A$, then the group G is $\overline{HA_p}$-group and $N_G^A = N_G$. If the norm N_G^A is non-Abelian, then following theorem takes place.

Theorem 1. *Finite p-groups ($p \neq 2$) G with non-Abelian norm N_G^A of Abelian non-cyclic subgroups are groups of the following types:*

1) G is $\overline{HA_p}$-group, $G = N_G^A = N_G$;

2) $G = (\langle x\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$, where $|x| = p^n$, $n > 1$, $|b| = |c| = p$, $[x,b] = 1$, $[b,c] = x^{p^{n-1}}$, $[x,c] = x^{p^{n-1}\alpha} b^{\beta}$, $(\beta, p) = 1$, $N_G^A = N_G = (\langle x^p\rangle \times \langle b\rangle)\langle c\rangle$;

3) $G = \langle x\rangle \langle b\rangle$, where $|x| = p^k$, $|b| = p^m$, $m > 2$, $k \geq m+r$, $Z(G) = \left\langle x^{p^{r+1}}\right\rangle \times \left\langle b^{p^{r+1}}\right\rangle$, $1 \leq r \leq m-1$, $[x,b] = x^{p^{k-r-1}s} b^{p^{m-1}t}$, $(s,p) = 1$, $N_G^A = N_G = \left\langle x^{p^r}\right\rangle \leftthreetimes \langle b\rangle$.

Taking into account the theorem 1 [1], author gets the next proposition.

Theorem 2. *If the norm N_G^A of Abelian non-cyclic subgroups of locally finite p-group ($p \neq 2$) G is non-Abelian, then $N_G^A = N_G$.*

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Neutral elements in n-ary hyperalgebras

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An *n-ary hyperalgebra* is a non-empty set H with the operation $f : H^n \to P(H)$, where $P(H)$ is the set of all non-empty subsets of H. An n-ary hyperalgebra (H, f) is an *n-ary hypersemigroup* if for every $i = 1, 2, \ldots, n$ it satisfies the identity

$$f(f(x_1^n), x_{n+1}^{2n+1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n+1}),$$

where x_i^j denotes the sequence $x_i, \ldots, x_j$. An n-ary hypersemigroup is an *n-ary hypergroup* if for all $a_1^n, b \in H$ and $i = 1, \ldots, n$ there exists $x_i \in H$ such that

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n). \tag{1}$$

An element $e \in H$ such that $x \in f(e, \ldots, e, x, e, \ldots, e)$ holds for every $x \in H$ is called a *weak identity*. There are n-ary hypergroups with two, three and more weak identities, and n-ary hypergroups without weak identities.

Theorem 1. *An n-ary hypersemigroup (H, f) is an n-ary hypergroup if and only if for all $a_1^n, b \in H$ the equation* (1) *is solved at the place $i = 1$ and $i = n$ or at the place $1 < i < n$.*

Theorem 2. *An n-ary hypersemigroup (H, f) is an n-ary hypergroup if and only if for all $a, b \in H$ there exists $x, y \in H$ such that*

$$b \in f(a, \dots, a, x) \cap f(y, a, \dots, a).$$

Theorem 3. *If an n-ary hypersemigroup (H, f) has a weak neutral element, then there exist a binary $(n = 2)$ hypersemigroup $(H, \circ)$ and a weak endomorphism φ such that*

$$f(x_1^n) \subseteq x_1 \circ \varphi(x_2) \circ \varphi(x_3) \circ \ldots \circ \varphi(x_{n-1}) \circ x_n$$

for all $x_1^n \in H$.

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On modular representations of Rees semigroups over cyclic group of order two

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Let $C_2 = \{e, a\}$ be a cyclic group of order two and P an $n \times m$ matrix with elements from $C_2 \cup \{0\}$ such that any row and column has at least one nonzero element. A Rees semigroup $\mathcal{R}(C_2, P)$ is a set of $m \times n$ matrices which contain exactly one nonzero element from C_2 with the following multiplication $A * B = APB$

Ponizovskii in the article [1] gave the criterion when Rees semigroup has finite and infinite representation type over a field which characteristic does not divide the order of the group. We are interested in representation type over a field of characteristic two. It is so called modular case.

The main result. Consider the semigroup $\mathcal{R}(C_2, P)$. Let F_2 be the two element field. It can be shown that a semigroup $\mathcal{R}(C_2, SPT)$ has the same representation type as the semigroup $\mathcal{R}(C_2, P)$ for any invertible matrices S, T over the group algebra $F_2[C_2]$. For the nonzero matrix P one can find matrices S, T such that SPT is a diagonal matrix, which has e, $e + a$ or 0 on the diagonal. Denote by r the number of e on the diagonal of the matrix SPT.

Theorem 1. *Let F be a field of characteristic two then following statements about representation type of $\mathcal{R}(C_2, P)$ over F hold*

- *finite type iff* $\max\{m - r, n - r\} = 1$ *and* $\min\{m - r, n - r\} = 0$*;*
- *tame infinite type iff* $m - r = n - r = 1$*;*
- *wild type iff* $\max\{m - r, n - r\} > 1$.

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Equivalence and factorization of partitioned matrices

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Let R be a principal ideal domain and $M(n, R)$ be a ring of $n \times n$ matrices over R. Let $T = \mathrm{triang}\{T_1, T_2, \ldots, T_k\}$ and $D = \mathrm{diag}\{D_1, D_2, \ldots, D_k\}$ be the cell-triangular and the cell-diagonal matrices with diagonal cells $T_i, D_i \in M(n_i, R), i = 1, 2, \ldots, k$. Matrices A and $B \in M(n, R)$ are called: a) right equivalent (right associate) if $B = AU$, where $U \in GL(n, R)$; b) equivalent if $B = PAQ$, where $P, Q \in GL(n, R)$.

In [1, 2] the connection between the equivalence of matrices T and D and the solving of linear matrix equations of Sylvesters type was established.

We consider the factorizations of the matrices T and D over R up to the association.

We established the conditions under which such partitioned matrices have up to the association the factorizations only of such corresponding partitioned form. We suggested also the method of construction of such factorizations through the factorizations of their diagonal cells and the solving of the linear matrix equations of Sylvesters type [4]. The necessary and sufficient conditions of uniqueness up to the association of the cell-triangular factorizations of the cell-triangular matrices over R that correspond to the factorizations of their diagonal cells are also established.

The nonsingular cell-diagonal matrices are right equivalent if and only if their corresponding diagonal cells are right equivalent. If the cell-diagonal matrices are singular, then the right equivalence of this matrices do not imply the right equivalence of their corresponding diagonal cells.

Theorem 1. *Let $A = \mathrm{diag}\{A_1, A_2, \ldots, A_k\}$ and $B = \mathrm{diag}\{B_1, B_2, \ldots, B_k\}$, $A_i, B_i \in M(n_i, R)$, $i = 1, \ldots, k$, be the singular cell-diagonal matrices.*

1) *Let among diagonal cells of matrices A and B be at most one singular cell A_j and B_j, respectively. Then the matrices A and B are right equivalent if and only if their corresponding diagonal cells are right equivalent.*

2) *Let matrices A and B have at least one nonsingular cell A_j and B_j, respectively, which are not right equivalent, then the matrices A and B are not right equivalent.*

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On a property of discrete sets in $\mathbb{R}^k$

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We consider discrete sets in $\mathbb{R}^k$ where each point has a finite multiplicity. We call them *discrete multiple sets*. For any two discrete multiple sets $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ we define a *distance* between them in such a way:

$$dist(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) = \inf_{\sigma} \sup_{n\in\mathbb{N}} |\, a_n - b_{\sigma(n)}|,$$

where infimum is taken over all bijections $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$. This function satisfies all the axioms of metric except the finiteness.

Definition 1. *A vector $\tau \in \mathbb{R}^k$ is called an ε-almost period of a discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$, if*

$$dist(\{a_n\}_{n\in\mathbb{N}}, \{a_n + \tau\}_{n\in\mathbb{N}}) < \varepsilon.$$

Definition 2. *A discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ is called almost periodic, if for each $\varepsilon > 0$ the set of its ε-almost periods is relatively dense in $\mathbb{R}^k$.*

A simple example of an almost periodic set is the set

$$\{\alpha m + F(m)\}_{m\in\mathbb{Z}^k}$$

where $F(m)$ is an almost periodic mapping from $\mathbb{Z}^k$ to $\mathbb{R}^k$, $\alpha > 0$.

For each almost periodic multiple set D there exists $M < \infty$ such that for all $c \in \mathbb{R}^k$ card $(D \cap \{x : \|x - c\| < 1) < M$. The limit of almost periodic multiple sets is almost periodic as well. For such sets some analogue of the Bochner criterion of almost periodicity holds. Any almost periodic multiple set D possesses finite nonzero shift invariant density

$$\Delta = \lim_{T\to\infty} \frac{\text{card } (D \cap \{x : |x_1| < T, \ldots, |x_k| < T\})}{(2T)^k}.$$

Next we prove that our definition of an almost periodic set is equivalent to the classical one: the discrete set is almost periodic if the measure with unit masses in points of the set is almost periodic in a weak sense.

An almost periodic set is one of models describing quasicristallic structures (see [1]).

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Measurable Cardinals and the Cofinality of the Symmetric Group

S. D. Friedman, L. Zdomskyy‡

A deep theorem of Macpherson and Neumann [3] states that if the symmetric group $Sym(\kappa)$ consisting of all permutations of a cardinal κ can be written as a union of an increasing chain $\langle G_i : i < \lambda \rangle$ of proper subgroups G_i, then $\lambda > \kappa$. The minimal λ with this property is denoted by $\mathrm{cf}(Sym(\kappa))$. It was proven in [4] that for every regular cardinals $\lambda > \kappa$ and a cardinal θ such that $\mathrm{cf}(\theta) \geq \lambda$, there exists a cardinal preserving forcing extension V^P such that $\mathrm{cf}(Sym(\kappa)) = \lambda$ and $2^\kappa = \theta$ in V^P. Moreover, for inaccessible κ we can assume that P is κ-directed closed. Therefore if κ is supercompact, then it remains so in V^P. Our main result states that the consistency of $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ at a measurable κ can be obtained assuming much less than supercompactness.

Theorem 1. *Suppose GCH holds and there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $(H(\kappa^{++}))^V = (H(\kappa^{++}))^M$. Then there exists a forcing extension V' of V such that κ is still measurable in V' and $V' \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$.*

The idea of the proof of Theorem 1 resembles that of the consistency of $\mathfrak{u} < \mathrm{cf}(Sym(\omega))$ established in [5]. In particular, we introduce a variant of Miller forcing and a (slightly more general than in [2]) variant of Sacks forcing at an inaccessible cardinal κ. Iterated forcing constructions where at each stage we take any of these forcing notions do not collapse κ^+. If such an iteration of length κ^{++} is suitably arranged, then $\mathrm{cf}(Sym(\kappa))$ is equal to κ^{++} in the corresponding forcing extension.

The idea of the proof that κ remains measurable after the reverse Easton iteration of the forcing notions described above can be traced back to the work [1], where the "tuning fork" argument was introduced.

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About linear cross isotopes of linear quasigroups

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It is well-known that every isotope of a quasigroup is a quasigroup, but it is not true for cross isotopes. Such a criterion has been found in [1].

Recall [2], that an operation f of the arity $n+1$ is called *i-cross isotopic of the type* $\vec{\imath} := (i_0, ..., i_s)$ or *cross isotopic of the type* $(m, \vec{\imath})$ to $(n+1)$-ary operation g, if $i_m = i \in \vec{\imath}$ is a subsequence of $(0, \ldots, n)$ and there exist a sequence $\vec{\alpha} := (\alpha_0, \ldots, \alpha_n, \alpha)$ of substitutions $\alpha_0, \ldots, \alpha_n, \alpha$ and an m-invertible operation h of the arity $s+1$ of the set Q such that

$$f(x_0, \ldots, x_n) = \alpha^{-1} g(\alpha_0 x_0, \ldots, \alpha_{i-1} x_{i-1}, \alpha_i h(x_{i_0}, \ldots, x_{i_s}), \alpha_{i+1} x_{i+1}, \ldots, \alpha_n x_n) \tag{1}$$

for all $x_0, \ldots, x_n \in Q$ and we will write $f = g(\vec{\alpha}, h)$ [3, 2].

A composition of translations and automorphisms of a group is called its *linear transformation.* If a multiary quasigroup is isotopic to a binary group and all components are its linear transformations, then the quasigroup is said to be *linear.* Every $(n+1)$-ary quasigroup g, being linear on the cyclic group $\mathbf{Z}_m$, can be given as follows [4]:

$$g(x_0, \ldots, x_n) = k_0 x_0 + k_1 x_1 + \cdots + k_n x_n + a \tag{2}$$

for some invertible elements $k_0, \ldots, k_n$ and an element a of the ring $\mathbf{Z}_m$. A cross isotopy $(\vec{\alpha}, h)$ is said to be *linear*, if α_0, ..., α_n, α, h are linear.

Let $\vec{\varepsilon} := (\varepsilon, \ldots, \varepsilon)$, where ε is the identity transformation of $\mathbf{Z}_m$, and let

$$h(x_{i_0}, \ldots, x_{i_s}) := \ell_0 x_{i_0} + \ell_1 x_{i_1} + \cdots + \ell_s x_{i_s} + b. \tag{3}$$

Theorem. *A linear i-cross ($i = i_m$) isotope $g(\vec{\varepsilon}; h)$ of the type $\vec{\imath} := (i_0, ..., i_s)$ of a linear quasigroup operation g (see (2)) is a quasigroup if and only if the elements $k_{i_m}\ell_p - \ell_m k_{i_p}$, $p = 0, 1, \ldots, s$, are invertible in the ring $\mathbf{Z}_m$.*

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Gaussian integers with divisors in narrow sectors

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Let $\mathbb{Z}[i]$ be the ring of the Gaussian integers. For $\varphi_1, \varphi_2 \in [0, \frac{\pi}{2}]$ we denote by $S = S(\varphi_1, \varphi_2)$ the sector

$$\left\{ \alpha \in \mathbb{Z}[i] \mid 0 \le \varphi_1 \le \arg \alpha < \varphi_2 \le \frac{\pi}{2} \right\}.$$

We define the arithmetic function on $\mathbb{Z}[i]$

$$\tau^{(S)}(\alpha) := \sum_{\substack{\delta \mid \alpha \\ \delta \in S}} 1.$$

Let

$$T(x;S) := \sum_{N(\alpha)\le x} \tau^{(S)}(\alpha).$$

It is obvious that $T(x;S)$ can be considered as the average order of the number of divisors in S of the Gaussian integer α for which $N(\alpha) \le x$.

Applying estimates of the second power moment of the Hecke zeta-function $Z(s,m)$ with the Grossen-character $\lambda_m(\alpha) = e^{4mi\arg\alpha}$ on the line $Res = \frac{1}{2}$ we obtain

Theorem. Let $\varphi_2 - \varphi_1 \gg x^{-\frac{1}{3}+\epsilon}$. Then uniformly at $S(\varphi_1,\varphi_2)$ the following asymptotic formula holds:

$$\sum_{N(\alpha)\le x} \tau^{(S)}(\alpha) = \frac{2(\varphi_2-\varphi_1)}{\pi}\,(Ax\log x + Bx) + O(x^{2/3+\epsilon}),$$

where A, B are the absolute constants, $A > 0$.

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Characterization of almost maximally almost-periodic groups

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For a topological group G, $G^\wedge$ denotes the group of all continuous characters on G endowed with the compact-open topology. Denote by $\mathbf{n}(G) = \cap_{\chi\in G^\wedge}\ker\chi$ the von Neumann radical of G. G. Lukács called a Hausdorff topological group G almost maximally almost-periodic if $\mathbf{n}(G)$ is non-trivial and finite and he raised the problem of describing them.

Following E.G.Zelenyuk and I.V.Protasov, we say that a sequence $\mathbf{u} = \{u_n\}$ in a group G is a *T-sequence* if there is a Hausdorff group topology on G for which u_n converges to zero. The group G equipped with the finest group topology with this property is denoted by $(G,\mathbf{u})$. Using the method of T-sequences, we give a general characterization of almost maximally almost-periodic groups.

Theorem 1. *Let G be an infinite group. Then the following statements are equivalent.*

1. *G admits a T-sequence $\mathbf{u}$ such that $\mathbf{n}(G,\mathbf{u})$ is non-trivial and finite.*
2. *G has a non-trivial finite subgroup.*

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Right Bezout rings with waists

A. Gatalevich

We study right Bezout rings with waists, that is, right Bezout rings where Jacobson radical contains completely prime ideal [1].

Kaplansky [2] defined the ring R to be an elementary divisor ring if every matrix A over R (not necessarily square) admits diagonal reduction, that is, there exist invertible matrices P and Q such that PAQ is diagonal matrix, say (d_{ij}), for which d_{ii} is a total divisor of $d_{i+1,i+1}$ for each i. He defined a ring R to be right Hermite ring if every 1×2 matrix over R admits diagonal reduction and showed that a right Hermite ring is a right Bezout ring, i.e., a ring for which every finitely generated right ideal is principally generated. For integral domains the notions of right Hermite and right Bezout are equivalent.

A ring R is a right distributive ring if its lattice of right ideals is distributive [3].

Theorem 1. *Let R be a right Bezout ring with waists. Then R is right Hermite ring.*

Theorem 2. *Let R be a distributive Bezout ring in which Jacobson radical contains completely prime ideal, and there exist no other two-sided ideals in R. Then R is not an elementary divisor ring.*

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The structure of minimal left ideals of the Superextensions of Abelian groups

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In the talk we shall discuss the structure of minimal (left) ideals of the superextensions $\lambda(G)$ of Abelian groups G. By definition, a family $\mathcal{L}$ of subsets of a set X is called a *linked system* on X if $A \cap B$ is nonempty for all $A, B \in \mathcal{L}$. Such a linked system is maximal linked if it coincides with any linked system M on X that contains $\mathcal{L}$. The space $\lambda(X)$ of all maximal linked systems on X is called the *superextension* of X. It is endowed with the topology generated by the sub-base consisting of the sets $U^+ = \{\mathcal{L} \in \lambda(X) : U \in \mathcal{L}\}$, where U runs over subsets of X.

It is known [7], [6] that each binary operation $*$ on X extends to a right topological operation on $\beta(X)$, the Stone-Cech compactification of X, playing a crucial role in Combinatorics of Numbers. In the same way the operation $*$ can be further extended to a right-topological operation on $\lambda(X)$ by the formula:

$$\mathcal{U} \circ \mathcal{V} = \Big\{ \bigcup_{x \in U} x * V_x : U \in \mathcal{U},\ \{V_x\}_{x \in U} \subset \mathcal{V} \Big\}.$$

If the operation $*$ on X is associative, then it extends to an associative operation on $\lambda(X)$. In this case $\beta(X)$ is a subsemigroup of $\lambda(X)$, see [1].

The interest to studying the semigroup $\lambda(X)$ was motivated by the fact that for each maximal linked system $\mathcal{L}$ on X and each partition $X = A \cup B$ of X into two sets A, B either A or B belongs to $\mathcal{L}$. This makes possible to apply maximal linked systems to combinatorial and Ramsey problems.

Theorem 1. *A maximal linked system $\mathcal{L}$ on a group G is a right zero of $\lambda(G)$ if and only if $\mathcal{L}$ is invariant in the sense that $xL \in \mathcal{L}$ for all $L \in \mathcal{L}$ and all $x \in G$.*

We define a group G to be *odd* if the order of each element x of G is odd. If G is a finite odd group, then the maximal linked system

$$\mathcal{L} = \{A \subset G : |A| > |G|/2\}$$

is invariant.

Theorem 2. *The superextension $\lambda(G)$ of a group G possesses a right zero if and only if G is odd.*

A maximal linked system $\mathcal{Z} \in \lambda(G)$ on a group G is invariant if and only if $\mathcal{Z}$ is a right zero of the semigroup $\lambda(G)$ if and only if the singleton $\{\mathcal{Z}\}$ is a minimal left ideal in $\lambda(G)$. Taking into account that the invariant maximal linked systems form a closed subsemigroup of right zeros of $\lambda(G)$, we obtain the following

Theorem 3. *A group G is odd if and only if all the minimal left ideals of $\lambda(G)$ are singletons. In this case the minimal ideal $K(\lambda(G))$ of $\lambda(G)$ is a closed right zeros semigroup consisting of invariant maximal linked systems.*

We recall that the group $\mathbb{Z}_2$ of integer 2-adic numbers is a totally disconnected compact metrizable Abelian group, which is the limit of the inverse sequence

$$\cdots \to C_{2^n} \to \cdots \to C_8 \to C_4 \to C_2$$

of cyclic 2-groups C_{2^n}.

By the continuity of the functor λ in the category of compact Hausdorff spaces, the superextension $\lambda(\mathbb{Z}_2)$ can be identified with the limit of the inverse sequence

$$\cdots \to \lambda(C_{2^n}) \to \cdots \to \lambda(C_8) \to \lambda(C_4) \to \lambda(C_2)$$

of finite semigroups $\lambda(C_{2^k})$. This implies that $\lambda(\mathbb{Z}_2)$ is a metrizable zero-dimensional compact topological semigroup.

Theorem 4. *Minimal left ideals of the semigroup $\lambda(\mathbb{Z})$ are compact metrizable topological semigroups homeomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$.*

In fact, the structure of minimal left ideals can be revealed with help of a faithful representation of the semigroup $\lambda(X)$ in the semigroup of all self-maps of the power-set of X.

Theorem 5. *For a finitely generated abelian group X the minimal left ideals of $\lambda(X)$ are compact metrizable topological semigroups, topologically isomorphic to countable products of cyclic 2-groups and finite cardinals endowed with the left zero multiplication.*

In particular, each minimal left ideal of $\lambda(\mathbb{Z})$ is topologically isomorphic to $2^\omega \times \prod_{k=1}^{\infty} C_{2^k}$, where the Cantor cube 2^ω is endowed with a left zero multiplication.

Theorem 6. *The superextension $\lambda(\mathbb{Z})$ contains a topological copy of each second countable profinite semigroup.*

This result contrasts with the famous Zelenuk's Theorem asserting that the semigroup of ultrafilters $\beta(X)$ contains no finite subgroup.

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Matrices that are self-congruent only via matrices of determinant one

Tatyana G. Gerasimova

This is joint work with Roger A. Horn and Vladimir V. Sergeichuk.

The following theorem was proved by Đocović and Szechtman [2, Corollary 4.7] based on Riehm's classification of bilinear forms [3].

Theorem 1. *Let M be a square matrix over a field $\mathbb{F}$ of characteristic different from 2. The following conditions are equivalent:*

(i) *for each nonsingular S, $S^TMS = M$ implies $\det S = 1$ (i.e., each isometry on the bilinear space over $\mathbb{F}$ with scalar product given by M has determinant 1),*

(ii) *M is not congruent to $A \oplus B$ with a square A of odd size.*

Coakley, Dopico, and Johnson [1, Corollary 4.10] gave another proof of this theorem for real and complex matrices only: they used Thompson's canonical pairs of symmetric and skew-symmetric matrices for congruence [4]. We give another proof of this theorem using the canonical matrices for congruence, constructed in [5]. For the complex field, canonical forms of 8 different types are employed in [1]; in contrast, our canonical forms are very simple and of three types.

We also describe all M satisfying (i) in terms of canonical forms of M for congruence, of (M^T, M) for equivalence, and of $M^{-T}M$ (if M is nonsingular) for similarity.

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Torsion Graph Over Multiplication Modules

Sh. Ghalandarzadeh, P. Malakooti Rad*

Let R be a commutative ring and M be an R-module. In this talk, we give a generalization of the concept of zero-divisor graph in a commutative ring with identity to torsion-graph in a module. We associate to M a graph denoted by $\Gamma(M)$ called torsion graph of M whose vertices are non-zero torsion elements (elements which have non-zero annihilator) of M and two different elements x, y are adjacent if and only if $[x : M][y : M]M = 0$. The residual of Rx by M, denoted by $[x : M]$, is a set of elements $r \in R$ such that $rM \subseteq Rx$ for $x \in M$. We investigate the interplay between module-theoretic properties of M and the graph-theoretic properties of $\Gamma(M)$. Let Γ be a graph and $V(\Gamma)$ denote the vertices set of Γ. Let $v \in V(\Gamma)$, then $w \in V(\Gamma)$ is called a complement of v, if v is adjacent to w and no vertex is adjacent to both v and w. Moreover, Γ is complemented if every vertex has a complement, and is uniquely complemented if it is complemented and any two complements of vertex are adjacent to the same vertices. An R-module M is a multiplication module if for every R-submodule K of M there is an ideal I of R such that $K = IM$. Among the other results, we prove the following:

Proposition 1. *Let R be a commutative ring and M be a multiplication R-module with $nil(M) \neq 0$, then*

(a) If $\Gamma(M)$ is complemented, then either $|M| \leq 14$ or $|M| \geq 15$ and $nil(M) = \{0, x\}$ for some $0 \neq x \in M$.

(b) If $\Gamma(M)$ is uniquely complemented and $|M| \geq 15$, then any complemented of the nonzero $x \in Nil(M)$ is an end.

Theorem 1. *Let R be a commutative ring and M be a faithful multiplication R-module with $nil(M) \neq 0$. If $\Gamma(M)$ is a uniquely complemented graph then, either $\Gamma(M)$ is star graph with at most five edges or $\Gamma(M)$ is an infinite star graph with center x, where $Nil(M) = \{0, x\}$.*

Theorem 2. *Let R be a commutative ring and M be a multiplication R-module. If $\Gamma(M)$ is complemented, but not uniquely complemented then, $M = M_1 \oplus M_2$, where M_1, M_2 are submodules of M.*

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Weakly Prime submodule

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Let R be a commutative ring with identity and M be an unitary R-module. In this paper we investigate weakly prime submodule. A proper submodule N of a module M over a ring R is said to be weakly prime if $IJK \subseteq N$ for $I, J \trianglelefteq R$ and $K \leq M$ implies that either $IK \subseteq N$ or $JK \subseteq N$, also we know that a proper submodule N of a module M is said to be semiprime if $I^2K \subseteq N$ for $I \trianglelefteq R$ and $K \leq M$ implies that $IK \subseteq N$, it is clear that every prime submodule is semiprime submodule. It is also obvious that each prime submodule is weakly prime but not conversely. A nonempty set $S \subseteq M$ is called m-system (weakly m-system) in M if for every $\alpha, \beta \in S$ there exists $r \in R$ such that $[\alpha : M]r[\beta : M]M \subseteq S$ ($[\alpha : M]r[\beta : M]M \cap S \neq \emptyset$). One of the fundamental cornerstones of commutative ring theory is the "prime avoidance" theorem, we proved an extension of the following:

Theorem 1. *Let N be a R-module and $H_1, \ldots, H_{n-2}$ be weakly prime submodules of M and H_{n-1}, H_n, N be submodules of M and N be contained in $\cup_{i=1}^{n} H_i$, then $N \subseteq H_i$ for some $i \in \{1, \ldots, n\}$.*

For any submodule N of M let

(1) $\sqrt{N} := \{u \in M|$ every m-system containing u meets $N\}$.
(2) $V(N) = \{H \in Spec(M)|N \subseteq H\}$

Theorem 2. *Let N be a submodule of M, Then:*

(1) $\sqrt{N} = \bigcap_{H \in V(N)} H$.
(2) If $V(N) = \emptyset$, *then* $\sqrt{N} = M$.

Theorem 3. *Let N be a weakly prime submodule of M, the following are equivalent:*

(1) N is a semiprime submodule.
(2) N is an intersection of weakly prime submodules.
(3) $N = \sqrt{N}$.

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Fibonacci length and special automorphisms of $(l, m|n, k)$-group

R. Golamie

The $(l, m|n, k)$-groups have been considered by M.Ejvet and R.M.Thomas [1]. In this paper, by studying the Fibonacci length of the class of the finitely presented parametric groups $(l, m|n, k)$ defined by $< a, b | a^l = b^m = (ab)^n = (ab^{-1})^k = 1 >$, we prove that there are some different subclasses of the same Fibonacci lengths. These lengths are independent of one of the parameters and involve the Wall number $\kappa(n)$. Moreover, the Fibonacci lengths of two homomorphic images of these groups have been calculated and compared with those of these groups. And also we will give the special automorphism for every class of the groups.

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Cayley graphs, dessins d'enfants and modular curves

K. Golubev

Maps (i.e. graphs drawn without intersecting edges) on surfaces are parameterized by conjugacy classes of subgroups of the so-called *Grothendieck oriented cartographic group*, which may be defined by the following presentation

$$\mathcal{C}_2^+ = \langle \rho_0, \rho_1, \rho_2 \,|\, \rho_1^2 = \rho_2\rho_1\rho_0 = 1\rangle$$

Modular group is isomorphic to the factor-group of $\mathcal{C}_2^+$:

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle \rho_0, \rho_1, \rho_2 \,|\, \rho_1^2 = \rho_2\rho_1\rho_0 = 1, \rho_0^3 = 1\rangle \simeq \mathcal{C}_2^+/\langle\, \rho_0^3 = 1\rangle.$$

Definition 1. *Extended modular group is the group presented as follows*

$$\mathrm{EPSL}_2(\mathbb{Z}) = \langle \Sigma \,|\, \sigma_i^2 = 1, i = 1, 2, 3; (\sigma_0\sigma_2)^2 = 1 = (\sigma_1\sigma_2)^3\rangle.$$

The group $\mathrm{PSL}_2(\mathbb{Z})$ can be embedded in $\mathrm{EPSL}_2(\mathbb{Z})$ as index 2 subgroup of words of even length in the alphabet Σ. Defining generating set Σ as transformations of the complex upper halfplane $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ as follows

$$\Sigma = \{\, \sigma_0(\tau) = -\overline{\tau},\, \sigma_1(\tau) = -\overline{\tau} + 1,\, \sigma_2(\tau) = \frac{1}{\overline{\tau}}\,\}$$

induces action of the extended modular group $\mathrm{EPSL}_2(\mathbb{Z})$ on $\mathbb{H}$. Fundamental domain of this action is a geodesic triangle with vertices in i, $e^{\frac{\pi i}{3}}$ and ∞.

Definition 2. *Universal dessin d'enfant $Dess_\infty$ is a graph on $\mathbb{H}$ obtained by acting on boundary of the* $\mathrm{EPSL}_2(\mathbb{Z})$ *fundamental domain with the group* $\mathrm{EPSL}_2(\mathbb{Z})$.

Theorem 1. *The graph dual to the graph $Dess_\infty$ is isomorphic to Cayley graph $Cal(\mathrm{EPSL}_2(\mathbb{Z}), \Sigma)$.*

Corollary 1. *Let $G \lhd \mathrm{PSL}_2(\mathbb{Z})$ be normal subgroup of the modular group of the finite index. Then factor-dessin $\mathbb{H}Dess_\infty/G$ (i.e. dessin d'enfant on $\mathbb{H}/G$ induced by the universal dessin d'enfant) is dual to the Cayley graph of the factor-group $Cal(\mathrm{EPSL}_2(\mathbb{Z})/G, \Sigma)$.*

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On an algebra of languages representable by vertex-labeled graphs

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Recently directed vertex-labeled graphs have been successfully applied to the diverse areas of computer science, robotics, etc. [1,2]. The properties of languages that can be represented by these graphs were studied in [3]. In this work we introduce an algebra that may serve as a an effective tool for characterization of languages that can be represented by vertex-labeled graphs.

Let X^+ be the set of all non-empty words on a finite alphabet X. We consider an algebra $\langle 2^{X^+}, \circ, \cup, \circledast, \varnothing, X \rangle$, where the operations on languages $L, R \in 2^{X^+}$ are defined as follows.

1) $L \cup R = \{w | w \in L \text{ or } w \in R\}$;

2) $L \circ R = \{w_1 \circ w_2 | w_1 \in L \text{ and } w_2 \in R\}$. For all $w_1, w_2 \in X^+$ and $x, y \in X$ $w_1x \circ yw_2 = w_1xw_2$ if $x = y$, else $w_1x \circ yw_2$ is undefined.

3) $L^* = \bigcup_{i=0}^{\infty} L^i$, where $L^0 = X$; $L^{n+1} = L^n \circ L$, $n \geq 0$; 4)$L^{\circledast} = L_{beg} \circ L^* \circ L_{end}$, where $L_{beg} = \{x | xw \in L, x \in X, w \in X^*\}$; $L_{end} = \{x | wx \in L, x \in X, w \in X^*\}$.

According to the definition of regular expressions in Kleene algebra, we define a regular expressions in algebra $\langle 2^{X^+}, \circ, \cup, \circledast, \varnothing, X \rangle$ as follows:

1) $\varnothing, x, y, xy$ are regular expressions for all $x, y \in X$.

2) If L and R are regular expression, then $L \cup R$, $L \circ R$, $L^{\circledast}$ are regular expressions.

The language represented by regular expression R is denoted by L(R).

Let $G = (Q, E, X, \mu)$ be a directed vertex-labeled graph, where Q is a finite set of vertices, $E \subseteq Q \times Q$ is a set of directed edges, X is a finite set of labels, $\mu : Q \to X$ is a mapping from set of vertices to the set of labels. A finite sequence of vertices $l = q_1q_2...q_k$ such that $(q_i, q_{i+1}) \in E$ is a path in a graph G. By L(G) we denote a language generated from the graph G as the set of labels for all paths in a graph G that begins in an initial vertex and ends in a final one.

Theorem 1. *Let L be a language of X^+. The following conditions are equivalent:*
1)L is represented by a regular expression in algebra $\langle 2^{X^+}, \circ, \cup, \circledast, \varnothing, X \rangle$;
2)L is a language generated from the vertex-labeled graph.

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On minimal identifiers of the vertices of the vertex-labeled graphs

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Existence conditions for subsets of the labeled graph vertex language that are minimal in terms of quantity and word length and discriminate this vertex from any other vertex are considered.

Let $G = (G, E, M, \mu)$ be finite, simple, labeled graph with set of vertices G, set of edges E, set of labels M and surjective labeling function $\mu : G \to M$. The set $O(g) = E(g) \cup \{g\}$ is called the neighborhood of vertex g. Graph G is called deterministic if for any vertex g and any $s, t \in O(g)$, $s \neq t$ implies $\mu(s) \neq \mu(t)$. The sequence of vertices labels $\mu(g_1) \ldots \mu(g_k)$ corresponding to some path $g_1 \ldots g_k$ in graph G is called a word. L_g denotes a set of all words produced by vertex $g \in G$. A vertex identifier $g \in G$ is a finite set of words $W_g \subseteq M^+$ such that for any vertex $h \in G$ the equality $W_g \cap L_g = W_g \cap L_h$ holds iff $g = h$.

An identifier W_g is minimal, if any set obtained from W_g through deletion of a single word or replacement of any word with its proper initial segment, is not an identifier of g. It is demonstrated that the set of minimal identifiers is infinite in general. A vertex identifier of a strongly deterministic (SD) graph [1] is reduced if none of its words contains palindromic subwords and is not a proper initial segment of any other word belonging to this identifier. It is demonstrated that for every vertex identifier of an SD graph there exists a unique reduced identifier and a procedure for reduction is proposed. It is demonstrated that the set of minimal reduced identifiers is infinite in general case. A procedure for matching any set $W \subseteq M^+$ of words with same prefixes with a rooted tree $T(W)$ and a procedure for its determination are proposed. A traversal of a rooted tree $T(W)$ is any set of words corresponding to any set of paths that pass in the aggregate through all vertices of the tree.

Theorem 1. *If a set W is a minimal identifier of an SD graph vertex g, then any traversal of a rooted tree $T(W)$ is an identifier of g as well.*

It is demonstrated that for vertex identifiers of arbitrarily labeled oriented graphs and initial identifiers of states of finite automata [2] an analogous theorem does not hold in general.

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Relation $\mathcal{J}$ on finitary factorpowers of $S_{\mathbb{N}}$

S. V. Gudzenko

The construction of the factorpower $FP\,(S)$ of the semigroup of transformations (S, M) is defined as a quotient semigroup $P\,(S)/_{\sim_M}$ where $P\,(S)$ is the global oversemigroup of S and the congruence $\sim_M$ is given by the definition:

$$A \sim_M B \Leftrightarrow \forall m \in M\,(A(m) = B(m))\,.$$

Green's relations on the factorpower of the finite symmetric group were researched in [1]. In the infinite case we can also consider different finitary versions of factorpowers. In particular next variants are natural for the symmetric group of the denumerable order:

$$S_{fin}(\mathbb{N}) = \{\pi \in S_{\mathbb{N}} : |\,\{i : \pi(i) \neq i\}\,| < \infty\}\,, FP_{fin_0}\,(S_{\mathbb{N}}) = FP\,(S_{fin}(\mathbb{N}))\,;$$

$$FP_{fin_1}\,(S_{\mathbb{N}}) = \left\{A \in FP\,(S_{\mathbb{N}}) : \exists I \subseteq \mathbb{N}, |I| < \infty : \exists \{\sigma_i\}_{i\in I} \subseteq S_{\mathbb{N}} : A = \overline{\{\sigma_i\}_{i\in I}}\right\};$$

$$FP_{fin_2}\,(S_{\mathbb{N}}) = \left\{A \in FP\,(S_{\mathbb{N}}) : \sup_{i\in\mathbb{N}} |i^A| < \infty\right\};$$

$$FP_{fin_3}\,(S_{\mathbb{N}}) = \left\{A \in FP\,(S_{\mathbb{N}}) : \forall i \in \mathbb{N}\; |i^A| < \infty\right\}.$$

In [2] Green's relations $\mathcal{R}$ and $\mathcal{L}$ were described.

The report deals with Green's relation $\mathcal{J}$ on the finitary factorpowers of $S_{\mathbb{N}}$. The main result is

Theorem. *Let $FP_f\,(S_{\mathbb{N}})$ be one of the finitary factorpowers of $S_{\mathbb{N}}$ and $A, B \in FP_f\,(S_{\mathbb{N}})$. Then $A\mathcal{J}B \Leftrightarrow A = \sigma B \tau$ for some $\sigma, \tau \in S_{\mathbb{N}}$.*

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Semigroup Closures of Finite Rank Symmetric Inverse Semigroups

Oleg Gutik[†], Jimmie Lawson[‡] and Dušan Repovš[♯]

A partial one-to-one transformation on a set X is a one-to-one function with domain and range subsets of X (including the empty transformation with empty domain). There is a natural associative operation of composition on these transformations, $ab(x) = a(b(x))$ wherever defined, and the resulting semigroup is called the *symmetric inverse semigroup* $\mathcal{I}_X$ [2]. The symmetric inverse semigroup was introduced by Wagner [4]. If the domain is finite and has cardinality n, which is then also the cardinality of the range, the transformation is said to be of *rank* n. For each $n \geq 0$, the members of $\mathcal{I}_X$ of rank less than or equal to n form an ideal of $\mathcal{I}_X$, denoted $\mathcal{I}_\lambda^n$ if $|X| = \lambda$.

A subset D of a semigroup S is said to be *ω-unstable* if D is infinite and for any $a \in D$ and infinite subset $B \subseteq D$, we have $aB \cup Ba \nsubseteq D$. An *ideal series* for a semigroup S is a chain of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \subseteq I_m = S$. We call the ideal series *tight* if I_0 is a finite set and $D_k := I_k \setminus I_{k-1}$ is an ω-unstable subset for each $k = 1, \ldots, m$.

Recall that a semigroup S is a *semitopological semigroup* if it is equipped with a Hausdorff topology for which all left translation maps λ_s and all right translation maps ρ_s are continuous [3].

Proposition 1. *Let S be a semitopological regular semigroup that admits a tight ideal series $I_0 \subseteq \ldots, \subseteq I_m = S$. Then each I_k is closed in S and each member of $S \setminus I_{m-1}$ is an isolated point of S.*

Corollary 1. *Let $\lambda \geqslant \omega$ and let n be any positive integer. If τ is a topology on $\mathcal{I}_\lambda^n$ such that $(\mathcal{I}_\lambda^n, \tau)$ is a semitopological semigroup, then every element $\alpha \in \mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}$ is an isolated point of the topological space $(\mathcal{I}_\lambda^n, \tau)$.*

Proposition 2. *Let S be a semitopological inverse semigroup for which the inversion map $x \mapsto x'$ is continuous. If T is an inverse subsemigroup that admits a tight ideal series, then T is closed in S.*

Proposition 2 applies directly to the symmetric inverse semigroup $\mathcal{I}_\lambda^n$ for λ infinite and n a positive integer and yields the following corollary.

Corollary 2. *Let S be a semitopological inverse semigroup for which the inversion map $x \mapsto x'$ is continuous. If (an isomorphic copy of) $\mathcal{I}_\lambda^n$ is a subsemigroup of S, then it is a closed subset of S.*

Lemma 1. *The class of semigroups admitting a tight ideal series is closed under finite products.*

Lemma 2. *Let $h \colon S \to T$ be a surjective semigroup homomorphism such that each point inverse $h^{-1}(t)$ is finite. If S has a tight ideal series, then so does T.*

A *topological semigroup* is a Hausdorff topological space endowed with a continuous semigroup operation [1]. The next theorem is our main one on the non-existence of compact embeddings of certain $\mathcal{D}$-classes.

Theorem 1. *Let S be a topological semigroup and let T be a subsemigroup having a tight ideal series $I_0 \subseteq \ldots \subseteq I_m$. If $D := I_{k+1} \setminus I_k$ is a regular $\mathcal{D}$-class, then $\overline{D} = \mathrm{cl}_S(D)$ is not compact.*

Corollary 3. *For an infinite cardinal λ and positive integer n, if $\mathcal{I}_\lambda^n$ is a subsemigroup of a topological semigroup S, it cannot be the case that that $\mathrm{cl}_S(\mathcal{I}_\lambda^k \setminus \mathcal{I}_\lambda^{k-1})$ is compact for $1 \leq k \leq n$.*

Theorem 2. *For any infinite cardinal λ there exists no topology τ on $\mathcal{I}_\lambda^\infty := \bigcup_{n=1}^\infty \mathcal{I}_n$ such that $(\mathcal{I}_\lambda^\infty, \tau)$ is a compact semitopological semigroup.*

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Differential preradicals and differential preradical filters

O. L. Horbachuk[♯], M. Ya. Komarnytskyi[♯], Yu. P. Maturin[*]

Definition 1. *A differential preradical r of $A-DMod$ assigns to each differential A-module C its differential submodule $r(C)$ in such a way that for every differential A-homomorphism $f : N \to M$*

$$f(r(N)) \subseteq r(M).$$

Definition 2. *Let A be a differential ring with the derivation δ. A differential preradical filter of A is a collection F of differential left ideals of A possessing the following properties:*

DF1. $I \in F \& I \subseteq J \& J$ is a differential left ideal of $A \to J \in F$;

DF2. $I \in F \& a \in A \to (I : a^\infty) \in F$;

DF3. $I \in F \& J \in F \to I \cap J \in F$.

Definition 3. *Let A be a differential ring with the derivation δ. A differential radical filter of A is a differential preradical filter F of A possessing the following property:*

DF4. I is a differential left ideal of $A \& I \subseteq J \& J \in F \& (\forall a \in J : (I : a^\infty) \in F) \to I \in F$

Proposition 1. *Let A be a differential ring and B be a multiplicatively closed system of differential two-sided ideals, which are finitely generated as differential left ideals. Then the set $F_B = \{T|T$ is a differential left ideal of $A \& \exists L \in B : L \subseteq T\}$ is a differential radical filter of A.*

Proposition 2. *Let A be a differential ring and S be a differential two-sided ideal of A. If every differential left ideal of A is two-sided then the set*

$F_S = \{T|T$ is a differential left ideal of $A \& S + T = A\}$

is a differential radical filter of A.

Proposition 3. *Let A be a differential ring. If I is an idempotent ideal of A then*

$s : M \mapsto s(M)(s(M) = \{m \in M | \forall a \in I \forall n \in \{0, 1, 2, ...\} : m^{(n)}a = 0\})$ is a differential hereditary radical.

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Global dimension of polynomial rings in partially commuting variables

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Let $\mathcal{A}$ be any Abelian category. Define the *global dimension of* $\mathcal{A}$ by $\text{gl.dim}\,\mathcal{A} = \sup\{n \in \mathbb{N} : (\exists A, B \in \text{Ob } A)\ Ext^n(A, B) \neq 0\}$.

This work is devoted to the global dimension of the category of objects in an Abelian category with the action of a free partially commutative monoid.

For a ring R with 1, let $\text{gl.dim}\,R$ be the global dimension of the category of left R-modules. As it is well known, for the polynomial ring $R[x_1, \ldots, x_n]$ in pairwise commuting variables, the following equation holds

$$\text{gl.dim}\,R[x_1, \ldots, x_n] = n + \text{gl.dim}\,R\ .$$

Moreover, if $\mathcal{A}$ is any Abelian category with exact coproducts, then

$$\text{gl.dim}\,\mathcal{A}^{\mathbb{N}^n} = n + \text{gl.dim}\,\mathcal{A}$$

for the free commutative monoid $\mathbb{N}^n$ generated by n elements. We will present one of possible generalizations of this formula. Let $M(E, I)$ be a free partially commutative monoid generated by a set of variables E, where $I \subseteq E \times E$ is an irreflexive symmetric relation containing the pairs of commuting variables. We prove that

$$\text{gl.dim}\ \mathcal{A}^{M(E,I)} = n + \text{gl.dim}\ \mathcal{A}$$

for any Abelian category with exact coproducts where n is the sup of numbers of mutually commuting distinct elements of E. For example, if $R[M(E, I)]$ is the polynomial ring in variables $E = \{x_1, x_2, x_3, x_4\}$ for which $x_1x_2 = x_2x_1$, $x_2x_3 = x_3x_2$, $x_3x_4 = x_4x_3$, and $x_4x_1 = x_1x_4$, then for any ring R with 1 we have

$$\text{gl.dim}\,R[M(E, I)] = 2 + \text{gl.dim}\,R\ .$$

Moreover, we prove that for any projective free ring R and graded $R[M(E, I)]$-module A, there exits a resolution of A consisting of $m + n$ free graded $M(E, I)$-modules, where $m = \text{gl.dim}\,R$ and n is the sup of numbers of mutually commuting distinct elements of E.

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On centralizers of rational functions in the Lie algebra of derivations of $k[x, y]$

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Consider the Lie algebra $W_2(k) = Der(k[x, y])$ of derivations of the polynomial ring $k[x, y]$, where k is a field of characteristic zero. To describe the structure of the subalgebras of this algebra is an important and very difficult problem.

For a fixed polynomial $f \in k[x, y]$ we study the centralizer of f in $W_2(k)$, i. e., the Lie subalgebra $C_{W_2(k)}(f) = \{D \in W_2(k) \mid D(f) = 0\}$.

This Lie algebra corresponds to the stabilizer of the polynomial f in the group $Aut(k[x, y])$ of all automorphisms of $k[x, y]$. There is a natural $k[x, y]$-module structure on $C_{W_2(k)}(f)$.

Proposition 1. *$C_{W_2(k)}(f)$ is a free module of rank 1 over $k[x, y]$.*

Proposition 2. *Let D_0 be a free generator of $C_{W_2(k)}(f)$. Then the centralizer of an element $D = fD_0$ from $C_{W_2(k)}(f)$ is equal to the Lie subalgebra $(fk(h) \cap k[x, y]) \cdot D_0$ for some closed polynomial $h \in k[x, y]$.*
In particular it is a maximal abelian infinite dimensional subalgebra in $C_{W_2(k)}(f)$ and every maximal abelian subalgebra in $C_{W_2(k)}(f)$ is of this type.

Analogous results hold for the Lie algebra of derivations of the field of all rational functions in two variables $k(x, y)$.

We also study some derivations of the quadratic field extensions of $k(x, y)$, i. e., of

$$\mathbb{F} = k(x, y)[t], \quad t^2 = \xi, \quad \xi \in k(x, y).$$

Proposition 3. *Let $f \in k(x, y)$, let $\mathbb{F}$ be as above, and let $D = ad(f)$. Then the kernel of the derivation $D : \mathbb{F} \to \mathbb{F}$ equals $k[\bar{f}] + V_{\lambda_{f,\xi}}(D)$, where $V_{\lambda_{f,\xi}}(D) = \{u \in \mathbb{F} \mid D(u) = \lambda_{f,\xi} u\}$, $\lambda_{f,\xi} = -\dfrac{D(\xi)}{2\xi}$, and $\bar{f}$ is a generating rational function for f.*

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On Algorithms Inverting the Burau Representation

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The Burau representation ρ of the braid group B_n has been exploited for cryptography based on the braid group. In process of solving braid conjugacy problem using linear algebra methods, it is essential to know, how to recover preimage braids from the image of this representation. We propose the inverting algorithms for Burau representation, which are different from [3].

Let us recall that the n-braid group B_n can be presented by the $n-1$ Artin generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations $\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & if\ |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & if\ |i-j| = 1 \end{cases}$.

The Burau representation $\rho_B : B_n \to GL_n(\mathbb{Z}[x, x^{-1}])$ is defined by rule $\rho_B(\sigma_i) = diag(I_{i-1}, \begin{pmatrix} 1-x & x \\ 1 & 0 \end{pmatrix}, I_{n-i+1})$ for all $i \in \{1, \ldots, n-1\}$. This representation is known to be unfaithful for all $n \geq 5$. Images $A = \rho_B(w)$ of braids $w \in B_n$ are called the Burau matrices, which satisfy following conditions $\sum_{i=1}^{n} a_{ji} = 1$ and $\sum_{i=1}^{n} a_{ij}x^i = x^j$, for all $j \in \{1, \ldots, n\}$.

In [1] Hughes used algorithm inverting ρ_B for security analysis of the braid group cryptosystem. The main idea is to reconstruct w from $\rho_B(w)$ generator by generator from right to left by assuming that the first column with highest degree entry in $A = \rho_B(w)$ indicates a last generator of w. Lee and Park

[2] improved Hughes algorithm and proposed two new algorithms. These algorithms were compared [3] with respect to their success rate and elapsed time.

We proposed the algorithms inverting Burau representation of the submonoid B_n^+ of B_n and the braid group B_n. Our algorithms compute Artin generators of w from left to right and from both sides simultaneously. Using [4] we have obtained experimental results, which show that these algorithms have the higher success probabilities, but they are slower due to their self-correction process.

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Pointfree Representation of real lattice ordered linear map

Abolghasem Karimi Feizabadi

In classical topology it is proved, non constructively, that for a topological space X, every bounded Riesz map (real lattice ordered linear map) ϕ in $C(X)$ is of the form $\hat{x}$ for a point $x \in X$. In this paper our main objective is to give the pointfree version of this result. In fact, we constructively represent each real Riesz map on a compact frame M by prime elements.

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On classes of modules and preradicals defined by functor $Hom_R(U,\text{-})$

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Fixing a left R-module ${}_RU$ we consider the functor $H = Hom_R(U,\text{-}) : R\text{-}Mod \to \mathcal{A}b$ where $\mathcal{A}b$ is the category of abelian groups. The classes of R-modules, associated to this functor, and the preradicals of R-Mod defined by these classes are studied ([1]). In particular, the class $Gen\,({}_RU)$ of modules generated by ${}_RU$ defines an idempotent preradical r^U such that $\mathcal{R}(r^U) = Gen\,({}_RU)$, i.e. $r^U(M) = \sum\{Im\,f \,|\, f : U \to M\}$ for every $M \in R$-Mod.

The class of modules $Ker\,H = \{M \in R\text{-}Mod \,|\, H(M) = 0\}$ is torsion-free and defines an idempotent radical $\overline{r}^{\,U}$ such that $\mathcal{P}(\overline{r}^{\,U}) = Ker H$. Then $r^U \leq \overline{r}^{\,U}$ and $\overline{r}^{\,U}$is the least idempotent radical containing r^U. The condition is shown when $r^U = \overline{r}^{\,U}$ (the weak projectivity of ${}_RU$).

Acting by r^U to ${}_RR$ we obtain the ideal $I = r^U({}_RR)$ (the trace of ${}_RU$), which in his turn defines two pairs of preradicals (of diverse types): $\left(r^I, r^{(I)}\right)$ and $\left(r_I, r_{(I)}\right)$ ([2]). The properties of these preradicals are studied, as well as the relations between them and conditions for their coincidence.

Finally, the connections between preradicals $(r^U, \overline{r}^{\,U})$ and preradicals defined by I are investigated. In particular, we obtain the situation: $r^I \leq r^U \leq \overline{r}^{\,U}$ and $r^I \leq r^{(I)} \leq \overline{r}^{\,U}$. The criterion of coincidence of these four preradicals is shown.

These results represent the generalizations and modifications of some facts proved earlier in special conditions: for Morita contexts and adjoint situations ([3]).

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Quivers of finite rings

Nataliya Kaydan

We consider a finite rings with $1 \neq 0$. A finite ring A is indecomposable if A cannot be decomposed into a direct sum of two nonzero rings.

Let A be a finite ring with the radical R. Then A is a semiperfect finite ring and right (left) regular A-module $A_A({}_AA)$ has the following decomposition into a direct sum of indecomposable right (left) projective A-modules: $A_A = P_1^{n_1} \oplus ... \oplus P_s^{n_s}$ $({}_AA = Q_1^{n_1} \oplus ... \oplus Q_s^{n_s})$, where X^n is a direct sum of n copies of a module X.

Recall that a semiperfect ring A is reduced if $n_1 = n_2 = ... = n_s$.

Proposition. *If A is a reduced finite ring, then the quotient ring A/R is a direct product of s finite fields.*

We consider the right and the left quiver of a finite ring A [1]

Theorem 1. *A finite ring A is indecomposable if and only if its quiver $Q(A)$ is connected.*

Theorem 2. *The right and the left quivers of a finite commutative ring coincide.*

Theorem 3. *The quiver of a finite indecomposable commutative ring contains one vertex and some loops in this vertex.*

Theorem 4. *A finite ring A is uniserial if and only if its right quiver $Q(A)$ is either one vertex without loops or one loop.*

Denote by F_{p^n} a finite field with p^n elements.

Example. *Let* $A = \begin{pmatrix} F_2 & F_4 \\ 0 & F_4 \end{pmatrix}$.

The right quiver $Q(A)$ *is:* $Q(A) = \left\{ \begin{array}{ccc} 1 & & 2 \\ \bullet & \longrightarrow & \bullet \end{array} \right\}$ *and the left quiver* $Q'(A)$ *is:* $Q'(A) = \left\{ \begin{array}{ccc} 1 & & 2 \\ \bullet & \leftleftarrows & \bullet \end{array} \right\}$

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The Lattice of Fully Invariant Subgroups of Reduced Cotorsion Group

Tariel Kemoklidze

The report deals with questions of abelian group theory and the term *group* always means an additively written abelian group. The notation and terms, used in the report are taken from the monographs [1], [2].

p denotes a fixed prime number. tA is a torsion part of group A, quasicyclic p-group is denoted by $Z(p^\infty)$.

The investigation of the lattice of fully invariant subgroups of a group is an important task of the theory of abelian groups. Little is known about this issue concerning cotorsion groups. A group A is called cotorsion if any extension by a torsion free group C splits i.e. $Ext(C, A) = 0$. Reduced cotorsion group A can be represented in the following form:

$$A = \prod_p (T_p^\bullet \oplus C_p) \qquad (1)$$

where $tA = \oplus_p T_p$, $\quad T_p^\bullet = Ext(Z(p^\infty), T_p)$, and $C_p = Ext(Z(p^\infty), A/tA)$ is algebrically compact torsion free group (see [1,§54,55]).If in the equation (1) T_p is torsion complete group ([4]), or a direct sum of cyclic p-group (see [5]), then $T_p^\bullet$ and corresponding A groups are fully transitive and by means of indicators (see[2] §65,67) describe the lattice of fully invariant subgroups of group A. In the report group T_p is a countable direct sum of torsion complete p-groups, while cotorsion hull $T_p^\bullet$ is not fully transitive (see [3]), therefore to describe the mentioned lattice, lower semilattice $\overline{\Omega}$ will be built which is different from the set of indicators and the function $\Phi : A \to \overline{\Omega}$ is determined that satisfies the necessary conditions. Consequently we get that the lattice of fully invariant subgroups of groups A is isomorphic to the filters of semilattice $\overline{\Omega}$.

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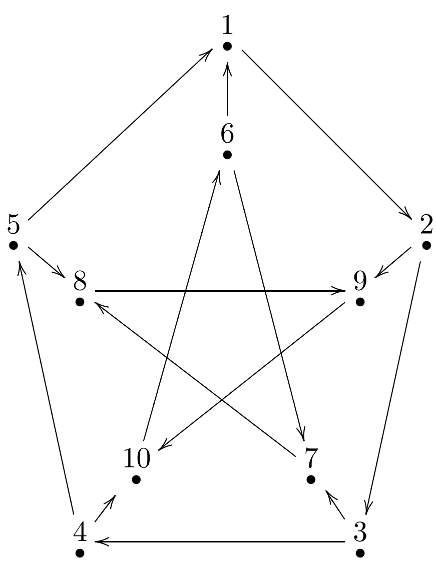
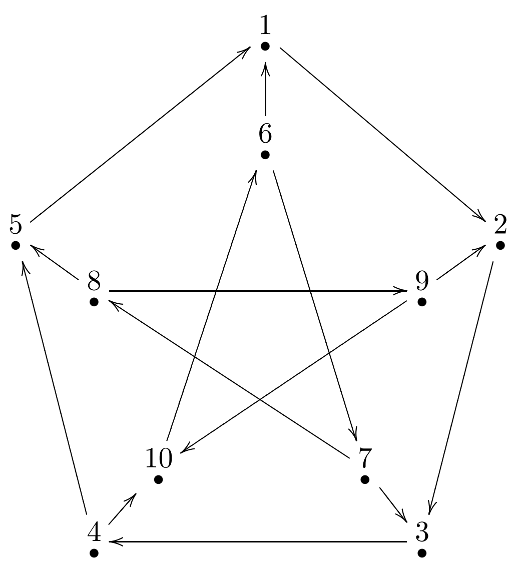
Orientations of the Petersen graph

Yu. Khomenko

Let P be the Petersen graph. Consider two orientations of Petersen graph:





Adjacency matrices of these quivers are:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

A characteristic polynomial of a quiver Q defined by formula $\chi_Q(x) = \chi_{[Q]}(x)$, where $[Q]$ is the adjacency matrix of Q. $\chi_{Q_1}(x) = \chi_{Q_2}(x) = x^{10} - 3x^5 - x^4 - x + 1$, but $Q_1 \not\simeq Q_2$.

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The isomorphism problem for finitary incidence rings

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The notion of a finitary incidence algebra was first introduced in [1] as a generalization of the notion of an incidence algebra for the case of an arbitrary poset. It was shown that the isomorphism problem for such algebras was solved positively ([1], Theorem 5). In the present talk we consider this problem in more general case.

Let $\mathcal{C}$ be a preadditive small category. Assume that the binary relation $\leq$ on the set of its objects, such that $x \leq y \iff Mor(x,y) \neq 0$, is a partial order. Consider the set of formal sums of the form

$$\alpha = \sum_{x \leq y} \alpha_{xy}[x,y], \tag{1}$$

where $\alpha_{xy} \in Mor(x,y)$, $[x,y]$ is a segment of the partial order. A formal sum (1) is called *a finitary series*, if for any $x, y \in Ob\,\mathcal{C}$, $x < y$ there exists only a finite number of $[u,v] \subset [x,y]$, such that $u < v$ and $\alpha_{uv} \neq 0$. The set of the finitary series is denoted by $FI(\mathcal{C})$.

The addition of the finitary series is inherited from the addition of the morphisms. The multiplication is defined by means of the convolution:

$$\alpha\beta = \sum_{x \leq y} \left(\sum_{x \leq z \leq y} \alpha_{xz}\alpha_{zy} \right) [x,y],$$

where $\alpha_{xz}\alpha_{zy} = \alpha_{zy} \circ \alpha_{xz} \in Mor(x,y)$. Under these operations $FI(\mathcal{C})$ form an associative ring with identity, which is called *a finitary incidence ring of a category.*

It is easy to see, that the description of the idempotents of $FI(\mathcal{C})$ can be obtained as in [1]. This allows us to solve the isomorphism problem for finitary incidence rings of preorders.

Let R be an associative ring with identity, $P(\preccurlyeq)$ an arbitrary preordered set. Denote by $\sim$ the equivalence relation on P, such that $x \sim y$ iff $x \preccurlyeq y$ and $y \preccurlyeq x$. Define $M([x],[y])$ to be an abelian group of row and column finite matrices over R, indexed by the elements of the equivalence classes $[x]$ and $[y]$. Consider the following preadditive category $\mathcal{C}$:

1. $Ob\,\mathcal{C} = P/_\sim$ with the induced order $\leq$;

2. For any pair $[x],[y] \in Ob\,\mathcal{C}$ define $Mor([x],[y]) = M([x],[y])$, if $[x] \leq [y]$, and 0 otherwise (the composition of the morphisms is the matrix multiplication).

Denote the finitary incidence ring of this category by $FI(P,R)$. Obviously, $FI(P,R)$ is an algebra over R, which is called *a finitary incidence algebra of P over R.*

Theorem 1. *Let P and Q be preordered sets, R and S indecomposable commutative rings with identity, $\mathcal{C}$ and $\mathcal{D}$ preadditive categories corresponding to the pairs (P, R) and (Q, S), respectively. If $FI(P, R) \cong FI(Q, S)$ as rings, then $\mathcal{C} \cong \mathcal{D}$.*

Corollary 1. *Let P and Q be class finite preordered sets, R and S indecomposable commutative rings with identity. If $FI(P, R) \cong FI(Q, S)$ as rings, then $P \cong Q$ and $R \cong S$.*

As a corollary we obtain the positive solution of the isomorphism problem for weak incidence algebras given in [2] .

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On Right Serial Quivers

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Let $Q = (VQ, AQ, s, e)$ be a quiver, which is given by two sets VQ, AQ and two mappings s, $e : AQ \to VQ$. The elements of VQ are called vertices or points and the elements of AQ are arrows. If an arrow $\sigma \in AQ$ connects the vertex $i \in VQ$ with the vertex $j \in VQ$, then $i = s(\sigma)$ is called its start vertex and $j = e(\sigma)$ is called its end vertex.

Recall that a right A-module is called serial if it decomposes into a direct sum of uniserial modules, that is, modules possessing a linear lattice of submodules.

A ring A is called right serial if A_A is a serial A-module.

A quiver Q is called right serial if each of its vertices is the starts of at most one arrow.

The quiver of right serial ring is right serial.

Let $N_n = \{1, \ldots, n\}$ and consider a map $\varphi : N_n \to N_n$. We represent φ as a right serial quiver Q_φ by drawing arrows from i to $\varphi(i)$ $(i \in N_n)$. A quiver Q_φ contains at least one oriented cycle.

Right serial quivers are considered, for example, in [1]–[3].

We discuss some properties of right serial quivers and its applications.

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Global dimension of semiperfect semidistributive prime rings

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We write SPSD-ring for a semiperfect semidistributive ring. A tiled order A is a prime right Noetherian SPSD-ring with the nonzero Jacobson radical. A tiled order A has the classical ring of fractions $M_n(D)$, where $M_n(D)$ is the ring of all $n \times n$-matrices with elements from division ring D.

We consider tiled orders of finite global dimension [1, §6.10]. Let A be a tiled order in $M_n(D)$, $1 = e_{11} + \ldots + e_{nn}$ be the decomposition of $1 \in M_n(D)$ into a sum usual matrix idempotents $e_{11}, \ldots, e_{nn} \in M_n(D)$.

Lemma. *There exists a tiled order B, which is isomorphic A and $1 \in B$ has the following decomposition*

$$1 = e_{11} + \ldots + e_{nn}, \text{ where } e_{11}, \ldots, e_{nn} \in B.$$

Theorem. *There are only finitely many tiled orders in $M_n(D)$ of finite global dimension with the following decomposition of $1 \in B$: $1 = e_{11} + \ldots + e_{nn}$.*

Corollary. *There are, up to isomorphism, only finitely many tiled orders in $M_n(D)$ of finite global dimension.*

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On the bands of semigroups

Aleksander Kizimenko

The Clifford's method of transitive homomorphism system is generalized for description of arbitrary sublattice of semigroup union.

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Classification of linear operators on a 5-dimensional unitary space

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The problem of classifying linear operators on a unitary space is a hopeless problem: it has the same complexity as the problem of classifying any number of linear operators on a unitary space [2]. Moreover, it contains the problem of classifying any system of linear operators on unitary spaces; that is, unitary representations of any quiver [4, 5].

Nevertheless, each matrix can be reduced to canonical form by Littlewood's algorithm [3] for reducing matrices to canonical form under unitary similarity. Various algorithms for reducing matrices to different canonical forms under unitary similarity were also proposed by Brenner, Mitchell, McRae, Radjavi, Benedetti and Gragnolini, Sergeichuk, and others; see Shapiro's survey [6].

For each n, Sergeichuk [4, 5] partitioned the infinite set of canonical n-by-n matrices under unitary similarity into a finite number of subsets consisting of matrices that can be obtained from the same parameter matrix. Using Littlewood's algorithm, we construct the list of all parameter matrices (with conditions on their parameters) that give canonical 5×5 matrices under unitary similarity.

This list was constructed twice. First it was obtained by direct calculations. Then we wrote a computer program that generate this list.

Note that invariants of linear operators on a 4-dimensional complex Euclidean space (more precisely, of 4-by-4 complex matrices under the action of $\mathrm{SO}_4(\mathbb{C})$ and $\mathrm{O}_4(\mathbb{C})$) were studied in [1].

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Completely isolated subsemigroups of wreath product of inverse symmetric semigroups

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For $n \in \mathbb{N}$ denote by $\mathcal{N}_n$ the set $\{1, 2, \dots, n\}$ and denote by $\mathcal{IS}_n$ finite inverse symmetric semigroup. Let $S^{P\mathcal{N}_n}$ be the set of partial functions from $\mathcal{N}_n$ to $\mathcal{IS}_m$

$$S^{P\mathcal{N}_n} = \{f : A \to \mathcal{IS}_m \,|\, \mathrm{dom}(f) = A, A \subseteq \mathcal{N}_n\}.$$

Given $f, g \in S^{P\mathcal{N}_n}$, the product fg is defined as

$$\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g), (fg)(x) = f(x)g(x) \text{ for all } x \in \mathrm{dom}(fg),$$

where $a \in \mathcal{N}_n, f \in S^{P\mathcal{N}_n}$, and f^a is defined as:

$$(f^a)(x) = f(x^a),\ \mathrm{dom}(f^a) = \{x \in \mathrm{dom}(a); x^a \in \mathrm{dom}(f)\}.$$

We consider a partial wreath product of inverse symmetric semigroups

$$\mathcal{IS}_m \wr_p \mathcal{IS}_n = \{(f, a) \in S^{P\mathcal{N}_n} \times \mathcal{IS}_n \,|\, \mathrm{dom}(f) = \mathrm{dom}(a)\},$$

with the product defined as $(f, a)(g, b) = (fg^a, ab)$.

If S is a semigroup and T is a subsemigroup of S, then T is called completely isolated if $ab \in T$ implies $a \in T$ or $b \in T$.

We show that the only completely isolated subsemigroups of semigroup $\mathcal{IS}_m \wr_p \mathcal{IS}_n$ are $\mathcal{IS}_m \wr_p \mathcal{IS}_n$, $\mathcal{S}_m \wr_p \mathcal{S}_n$, $(\mathcal{IS}_m \wr_p \mathcal{IS}_n) \setminus (\mathcal{S}_m \wr_p \mathcal{S}_n)$, $\mathcal{IS}_m \wr_p \mathcal{S}_n$, $\mathcal{IS}_m \wr_p (\mathcal{IS}_n \setminus \mathcal{S}_n)$, $(\mathcal{IS}_m \setminus \mathcal{S}_n) \wr_p \mathcal{S}_n$.

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Multiplication modules, in which every prime submodule is contained in unique maximal submodule

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Let R be associative ring with nonzero identity element. Left R-module M is called a multiplication module if for every submodule N of M, there exist an ideal B of R such that $N = BM$.

For a module M, *Spec*(M) denote the set of all prime submodules of M, $Max(M)$ denote the set of all maximal submodules of M. Two maximal submodules M_1 and M_2 of left module M is called to by related if $M/M_1 \cong M/M_2$.

Theorem 1. *Let M be any left multiplication module. The following conditions are equivalent:*

(1) Every prime submodule of M has, back to isomorphism, the unique simple homomorphic image;

(2) Every prime submodule of M is contained, back to relatednes, in unique maximal submodule of M;

A module M, satisfying these conditions is called lpm-module (left pm-module).

Theorem 2. *If M is multiplication left R-module, and Max(M) is retract of Spec(M), then M is lpm-module.*

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On commutative semigroups and splitting preradicals over them

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The category of acts over commutative semigroup S and the preradicals in the $S-Act$ are considered [1].

Recall a preradical $r : S - Act \to S - Act$ is called to be splitting, if for every $A \in S - Act$ we have $r(A) \amalg B = A$.

For example we proved that all preradicals over the commutative monoid S are splitting if and only if S is union of finite family of groups.

Some corollaries from this result were obtained.

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On 0-cohomology of completely 0-simple semigroups

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The general description of the 0-cohomology for a completely 0-simple semigroup $\mathcal{M}^0(G, I, \Lambda, P)$ was given in [1]. The main result in that paper is that $H_0^n(S, A) \cong H^n(G, A)$ for all $n > 2$. In the case when $n = 2$, which is mainly important for the applications, an exact sequence was found only. The aim of the following proposition is to give a more detailed description for the structure of 0-cocycles in $H_0^2(S, A)$.

Proposition 1. *Let $S = \mathcal{M}^0(G, I, \Lambda, P)$ be a completely 0-simple semigroup, A is a 0-module over S. Then any 0-cocycle $f \in Z_0^2(S, A)$ is equivalent to a 0-cocycle $\tilde{f} \in Z_0^2(S, A)$ which values do not depend on μ. More precisely,*

$$\tilde{f}(x_{i\lambda}, y_{j\mu}) = e_{i1}\xi(x, y, \lambda, j),$$

for some function $\xi(x, y, \lambda, j)$ which does not depend on i and μ.

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On fractal properties of some Cantor-type sets related to Q- representation of real numbers

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Let $s > 1$ be a fixed positive integer, $A = \{0, 1, \dots, s-1\}$, $Q = \{q_0, q_1, \dots, q_{s-1}\}$ be a set of numbers with properties:

$$\left.\begin{array}{ll} 1) & q_i > 0; \\ 2) & q_0 + q_1 + \ldots + q_{s-1} = 1 \ . \end{array}\right\} \qquad (1)$$

$\beta_0 = 0, \beta_i = q_0 + q_1 + \ldots + q_{j-1}$.

Theorem 1. *For any* $x \in [0,1]$*, there exists a sequence of numbers* $\alpha_k \in A$ *such that* $x = \beta_{\alpha_1} + \sum\limits_{k=2}^{\infty} \left[\beta_{\alpha_k} \prod_{j=1}^{k-1} q_{\alpha_j}\right]$ *(s-symbol Q-expansion of x).*

We denote this expression briefly by $\triangle_{\alpha_1 \dots \alpha_k \dots}$ (s-symbol Q-representation of x). The number is said to be the k-th Q-symbol of x.

In the talk we describe the fractal properties of the subset M of $[0,1]$ such that Q-representation has the following properties:
1. Any l-th $(1 < l \in N)$ Q-symbol of $x \in M$ is arbitrary.
2. Q-symbol with an ordinal number $n \notin \{1 + kl\}, k = 0, 1, 2, \dots$ is determined uniquely and it depends on all previous Q-symbols.

Let us consider a sequence of matrices:

$$\|\mathbf{c_{ij}^n}\| = \begin{pmatrix} c_{01}^n & c_{02}^n & \cdots & c_{0(l-1)}^n \\ c_{11}^n & c_{12}^n & \cdots & c_{1(l-1)}^n \\ \cdots & \cdots & \cdots & \cdots \\ c_{(s-1)1}^n & c_{(s-1)2}^n & \cdots & c_{(s-1)(l-1)}^n \end{pmatrix}, \quad n = 1, 2, \dots, \quad c_{ij}^n \in A = \{0, 1, \dots, s-1\}.$$

Theorem 2. *If the sequence* $||c_{ij}^n||_{n=1}^{\infty}$ *is a periodic sequence with a period*

$$\left(||c_{ij}^{n_1+1}||_{n=1}^{\infty}, \dots, ||c_{ij}^{n_1+2}||_{n=1}^{\infty}, ||c_{ij}^{n_1+p}||_{n=1}^{\infty}\right)$$

then the Hausdorff-Besicovitch dimension of the set M is the solution of the following equation

$$\sum_{i_1=0}^{s-1} \dots \sum_{i_p=0}^{s-1} \left(\prod_{k=1}^{p} \left[q_{i_k} \prod_{j=1}^{l-1} q_{c_{i_k j}^{n_1+k}} \right] \right)^x = 1.$$

Corollary 1. *If all matrices of the sequence* $||c_{ij}^n||_{n=1}^{\infty}$ *coincide, i.e.,* $c_{ij}^n = c_{ij}$*, then the Hausdorff-Besicovitch dimension of the set M is the solution of the following equation*

$$\sum_{i=0}^{s-1} \left[q_i \prod_{j=1}^{l-1} q_{c_{ij}}^n \right]^x = 1.$$

Corollary 2. *If $(c_{i1}\dots c_{i(l-1)}) = (c_{01}\dots c_{0(l-1)})$, $i=\overline{0,s-1}$, then the Hausdorff-Besicovitch dimension of the set M is the solution of the following equation*

$$(q_0^x + \ldots + q_0^{s-1}) \prod_{j=1}^{l-1} q_{c_{0j}}^x = 1.$$

Corollary 3. *If $q_i = \frac{1}{s}$, then the Hausdorff-Besicovitch dimension of the set M is equal to $\alpha_0(M) = \frac{1}{l} log_s(s-1)$.*

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On generalization of the theorem of Ramachandra

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Let $S_1^{(m)}$, $S_2^{(m)}$, $S_3^{(m)}$ be the sets of the Hecke zeta-functions $Z(as+b;m)$ with Grossencharacter $\lambda_m(\alpha) = e^{4mi\arg\alpha}$, $a,b\in\mathbb{R}$, m is fixed from $\mathbb{Z}$, derivatives, and the logarithms of z-functions, respectively. The $\log Z(as+b;m)$ is defined by analytic continuation from the half-plane $\sigma = \Re s > 1$.
Let

$$P_\ell^{(m)}(s) = \prod_{j=1}^{h} f_j^{z_j}(s), \;\; f_j \in S_\ell^{(m)}, \;\; z_j \in \begin{cases} \mathbb{C} & if\ \ell = 1; \\ \mathbb{Z} & if\ \ell = 2,3. \end{cases}$$

Furthermore, let

$$F^{(m)}(s) = P_1^{(m)}(s) P_2^{(m)}(s) P_3^{(m)}(s) F_0^{(m)}(s),$$

where

$$F^{(m)}(s) = \sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\neq 0}} \frac{g(\alpha)}{N(\alpha)^s} e^{4mi\arg\alpha},$$

$$F_0^{(m)}(s) = \sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\neq 0}} \frac{C_m(\alpha)}{N(\alpha)^s}, \;\; |C_m(\alpha)| \ll N(\alpha)^\varepsilon, \;\; \varepsilon > 0.$$

$$\sum_\alpha \frac{|C_m(\alpha)|}{N(\alpha)^\sigma} < \infty, \;\; for\ \sigma > \frac{1}{2}.$$

We prove the generalization of the theorem of Ramachandra[1]:

Theorem 1. *Let $N_M(\sigma,T)$ denote the number of zeros of all zeta-functions $Z(s,m)$, $|m|\le M$ in the rectangle $\sigma \le \Re s \le 1$, $|\Im s| \le T$, and let B be the constant for which*

$$N_M(\sigma,T) \ll (MT)^{B(1-\sigma)} (\log (MT)^2)$$

Then for $h \le x$, $0 \le \phi_1 < \phi_2 \le \frac{\pi}{2}$, $\phi_2 - \phi_1 \gg \exp(-c(\log x) + (\log\log x)^{-1})$ we have

$$S(x,h;\phi_1,\phi_2) := \sum_{\substack{x<N(\alpha)\le x+h\\ \phi_1<\arg\alpha\le\phi_2}} g(\alpha) = \frac{2\phi}{\pi} I(x,h) + R(x,h,\phi) \tag{1}$$

where

$$I(x,h)=\frac{1}{2\pi i}\int\limits_0^h\left(\int_{C_0(r)}F^{(0)}(s)(v+x)^{s-1}ds\right)dv$$

$$R(x,h,\phi)\ll x^{1-\frac{1}{B}+\varepsilon}+h\exp\left(-c(\log x)^{\frac{1}{3}}(\log\log x)^{-1}\right)$$

(here $r=c(\log x)^{-\frac{2}{5}}(\log\log x)^{-1}$, and $C_0(r)$ is the contour by starting from the circle $\{s\in\mathbb{C},\ |s-1|=r\}$, removing the point $s=1-r$, proceeding on the remaining portion of the circle in the anticlockwise direction).

From the estimates of the number of zeros of the Hecke function $Z(s,m)$ we infer that $B=\frac{5}{2}$ if $M\sim T$. We apply the relation (1) for constructing the asymptotic formulas of the summatory functions of type $z^{\omega(\alpha)}f(\alpha)$, where $\omega(\alpha)$ denotes the number of distinct prime divisors of α in $\mathbb{Z}[i]$, and $f(\alpha)$ belongs to special class of multiplicative functions.

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Characterizations of finite soluble and supersoluble groups

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All groups considered are finite. Let $\mathcal{X}$ be a class of groups. Then the symbol $G_{\mathcal{X}}$ denotes the product of all normal subgroups E of G such that $E\in\mathcal{X}$. Let A, K and H be subgroups of a group G and $K\leq H\leq G$. Then we say: (i) *A $\mathcal{X}$-conditionally covers or avoids the pair* (K,H) if there is an element $h\in H_{\mathcal{X}}$ such that either $AH=AK^h$ or $A\cap H=A\cap K^h$. A pair (K,H) is called *maximal* if K is a maximal subgroup of H. We use $\mathcal{S}$ to denote the class of all soluble groups. $\mathcal{G}$ is the class of all groups.

Theorem 1. *A group G is soluble if and only if G has a composition series $1=G_0<G_1<\ldots<G_n=G$ such that every Sylow subgroup of G $\mathcal{S}$-conditionally covers or avoids each maximal pair (K,H) of G with $G_{i-1}\leq K<H\leq G_i$ for some i.*

Theorem 2. *A group G is soluble if and only if G has a composition series $1=G_0<G_1<\ldots<G_n=G$ such that every maximal subgroup of G $\mathcal{S}$-conditionally covers or avoids each maximal pair (K,H) of G with $G_{i-1}\leq K<H\leq G_i$ for some i.*

Theorem 3. *Let G be a group. Then the following statements are equivalent:*

(1) G is supersoluble.

(2) Every subgroup of G $\mathcal{G}$-conditionally covers or avoids each maximal pair of G.

(3) Every $\cap$-irreducible subgroup of G $\mathcal{G}$-conditionally covers or avoids each maximal pair of G.

(4) Every cyclic subgroup of G with prime order and order 4 $\mathcal{G}$-conditionally covers or avoids each maximal pair of G.

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The width of verbal subgroups of unitriangular matrices groups over ring of $\mathbb{Z}/m\mathbb{Z}$

V. Kovdrysh

Let m be a positive integer and let $\mathbb{Z}/m\mathbb{Z}$ be a ring of congruence classes modulo m. Let $UT_n(\mathbb{Z}/m\mathbb{Z})$ denote the unitriangular group of degree n over ring $\mathbb{Z}/m\mathbb{Z}$ consisting of all unitriangular matrices

$$A = e + \sum_{1 \le i < j \le n} \alpha_{ij} e_{ij}, \alpha_{ij} \in \mathbb{Z}/m\mathbb{Z},$$

where e denotes the unit matrix of size n and e_{ij} is the matrix having identity in the place (i, j) and zeros elsewhere ([1]).

For a set of words $W = \{w_i\}_{i \in I}$ and a group G we define the verbal subgroup $V(G)$ of group G as the subgroup generated by all values of the words $w_i, i \in I$ in group G. We define the width of verbal subgroup $V(G)$ as a minimal positive integer k, such that every element from $V(G)$ is a product of at least k values of the words in group G. If such k does not exist, then we say that group G has a width ∞.

Theorem 1. *Let m and n be positive integer, then:*
a) the width of each element of lower central series of group $UT_n(\mathbb{Z}/m\mathbb{Z})$ is 1;
b) the width of each element of commutant series of group $UT_n(\mathbb{Z}/m\mathbb{Z})$ is 1.

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Posets which are acts over semilattices

I. B. Kozhukhov

An act over a semigroup S is a set X with the mapping $X \times S \to X$, $(x, s) \mapsto xs$ such that $x(st) = (xs)t$ for all $x \in X$, $s, t \in S$. A semilattice is a partially ordered set (poset) A in which every subset $\{a, b\} \subseteq A$ has the infimum $\inf\{a, b\}$. It is well known that the semilattice is exactly the commutative idempotent semigroup (with the multiplication $a \cdot b = \inf\{a, b\}$). It is not difficult to check that every act X over a semilattice S is a poset with the ordering $x \le y \Leftrightarrow x \in yS^1$. It is natural to ask, which posets can be acts over semilattices? It is proved in [1] that a connected poset X is an act over a chain iff the set $x^\nabla = \{y | y \le x\}$ is a chain for every $x \in X$ (the connectedness is here equivalent to the condition $\forall x, y\ \exists z \le x, y$). It is not difficult to prove that, if X is an act over a semilattice then the following condition holds: (A) $\forall x \in X\ (x^\nabla\ \mathit{is\,a\,semilattice})$.

Consider the following conditions on the map $\alpha : X \to X$: (1) α is isotone, i.e., $\forall x, y\,(x \le y \Rightarrow x\alpha \le y\alpha)$; (2) α is decreasing, i.e. $\forall x\,(x\alpha \le x$; (3) α is idempotent, i.e. $\alpha^2 = \alpha$; (4) $\forall x, y\,(x = x\alpha\,\&\,y \le x \Rightarrow y = y\alpha)$.

Proposition 1. *Let X be a poset, and Φ be the set of all maps satisfying conditions* (1)–(4). *Then Φ is an idempotent commutative semigroup (a semilattice).*

It is clear that the poset X is an act over a semilattice iff Φ acts on X transitively, i.e., $\forall x, y\,(x \le y \Rightarrow \exists \varphi \in \Phi\,(x = y\varphi))$. However it doesn't hold for all posets satisfying (A). For example, for the set $X = \{1, 2, 3, 4, 5, 6, 7\}$ where $7 < 5, 6$, $6 < 3, 4$, $5 < 1, 3$, $4 < 1, 2$, $3 < 1, 2$, there is no $\varphi \in \Phi$ such that $2\,\varphi = 3$.

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Checking experiments on automata in quasimanifolds

V. A. Kozlovskii

While studying checking experiments on automata different assumptions are made regarding behavior of reference automata and classes with respect to which experiments are considered [1]. Experiments constructed under restrictions of a given class F normally have greater distinguishing capacity then is imposed by F, i.e., they are checking experiments with respect to a wider class of automata. A maximal possible extension of class F characterizes maximal distinguishing capability. It is demonstrated that for traditionally considered class F_n of all automata with number of states equal to n such extension leads to classes that are quasimanifolds of automata.

Let us consider Mealy automata $A = (S, X, Y, \delta, \lambda)$, where S, X, Y are correspondingly the set of states, inputs and outputs, δ is transition function and λ is output function. Input-output word $w = (p, q)$ is generated by automaton A in state s, if $\lambda(s, p) = q$. A checking experiment for (A, F) is a set of input-output words W generated by automaton A in a certain state such that if W is generated by automaton $B \in F$ then B contains a subautomaton equivalent to A. Let reference automaton A be a DD-1 automaton, i.e., any input-output word generated by the state s of A is an initial identifier of this state [2]. Using graph-theoretic characterizations of automata representations [2] the following theorem is proved.

Theorem 1. *There exists a finite number of quasimanifolds $K_1, K_2, ..., K_f$ such that 1) an experiment W is a checking experiment for (A, F_n) iff W is a checking experiment for $(A, \cup_{i=1}^{f} K_i)$; 2) number of such quasimanifolds $f \geq 2^m$ if $n \geq 2^m$, where $m = |X|$; 3)$K(I_A^1) = \cap_{i=1}^{f} K_i$; 4) class $\cup_{i=1}^{f} K_i$ can not be extended without losing the property of conserving the set of checking experiments for (A, F_n).*

This theorem demonstrates that class extension under restrictions of conserving the set of checking experiments could have a rather complicated structure. This structure determines the complexity of checking experiments recognition problem. It is demonstrated that in the case considered this problem is NP-complete.

Analogous results hold for some other classes of automata, for example, for group and lossless automata.

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Про розв'язання загальних функційних рівнянь, подібних до Муфанг, над квазігрупами

Галина Крайнічук

Дані тези є продовженням тез [1], де зазначено, що всі загальні функційні рівняння, які мають по дві появи двох предметних змінних, а третя — чотири появи (такими, зокрема, є функційні рівняння Муфанг), над множиною квазігрупових операцій парастрофно рівносильні восьми функційним рівнянням. Нами знайдено розв'язки всіх цих рівнянь, крім рівняння Муфанг. Розв'язок одного рівняння подано в [1], розв'язки ще трьох рівнянь наведено в наступних теоремах.

Теорема 1 *Шістка квазігрупових операцій* $(f_1, \ldots, f_6)$, *що визначені на множині* Q, *є розв'язком функційного рівняння*

$$F_1(F_2(x;y); F_3(x;z)) = F_4(F_5(x;y); F_6(x;z)) \tag{1}$$

тоді і тільки тоді, коли для довільного елемента $e \in Q$ *існує лупа* $(Q;\cdot)$ *з нейтральним елементом* e, *підстановки* α, β, γ, δ *та відображення* ν *множини* Q *в середнє ядро лупи* $(Q;\cdot)$ *такі, що виконуються рівності*

$$f_1(x;y) = \alpha x \cdot \beta y, \quad f_2(x;y) = \alpha^{-1}(\gamma f_5(x;y) \cdot \nu x),$$
$$f_4(y;z) = \gamma y \cdot \delta z, \quad f_6(x;z) = \delta^{-1}(\nu x \cdot \beta f_3(x;z)).$$

Теорема 2 *Шістка квазігрупових операцій* $(f_1, \ldots, f_6)$, *що визначені на множині* Q, *є розв'язком функційного рівняння*

$$F_1(x; F_2(y; F_3(x;y))) = F_4(x; F_5(z; F_6(x;z))) \tag{2}$$

тоді і тільки тоді, коли існують підстановки α *та* γ, *такі, що виконуються рівності*

$$f_1(x;\gamma x) = f_4(x;\alpha x), \quad f_3(x;y) = f_2^r(y;\gamma x), \quad f_6(x;z) = f_5^r(z;\alpha x).$$

Теорема 3 *Шістка квазігрупових операцій* $(f_1, \ldots, f_6)$, *що визначені на множині* Q, *є розв'язком функційного рівняння*

$$F_1(F_2(x;y); F_3(x;y)) = F_4(F_5(x;z); F_6(x;z)) \tag{3}$$

тоді і тільки тоді, коли існує перетворення δ, *таке, що виконуються рівності*

$$f_2(x;y) = f_1^\ell(\delta x; f_3(x;y)), \quad f_5(x;z) = f_4^\ell(\delta x; f_6(x;z)).$$

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Left transitive quasigroups

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Quasigroup $Q\,(\cdot)$ is called left-transitive, if in $Q\,(\cdot)$ the identity is $xy \cdot xz = yz$ is true [1].
The following results are obtained:

1. Any left-transitive quasigroup $Q\,(\cdot)$ has left unit f, $f\,x = x$, $\forall\, x \in Q$.
2. Any left-transitive quasigroup $Q\,(\cdot)$ is an LIP-quasigroup.
3. Any loop which is an isotope of a left-transitive quasigroup $Q\,(\cdot)$ is a group.
4. Every left-transitive quasigroup $Q\,(\cdot)$ is a left Bol quasigroup, i.e. in $Q\,(\cdot)$ the identity
$$x\,(y \cdot xz) = R_{e_x}^{-1}\,(x \cdot yx) \cdot z$$
holds.
5. A left-transitive quasigroup $Q\,(\cdot)$ is isotopic to an abelian group in only case, when R_f is a quasigroup automorphism.
6. Left nucleus N_l of a left-transitive quasigroup $Q\,(\cdot)$ has the form
$$N_l = \{a \in Q \mid af = a\,,\;\; ax = xf \cdot a,\;\; \forall\, x \in Q\}\,.$$
7. Any quasiautomorphism γ of a left-transitive quasigroup $Q\,(\cdot)$ has the form $\gamma = R_k R_f \gamma_0$, where γ_0 is an automorphism of quasigroup $Q\,(\cdot)$, k is some fixed element of set Q.
8. In any left-transitive quasigroup $Q\,(\cdot)$, any quasiautomorphism γ is an automorphism of quasigroup $Q\,(\cdot)$ in only case, when $\gamma\,f = f$.
9. Examples are constructed.

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The signature operator for graded Hilbert subdifferential Hodge spaces

Jan Kubarski

There is the well known Hirzebruch signature operator for smooth manifolds and for Lipschitz manifolds (N.Teleman). The crucial role plays the Hodge theory and the Poincare duality property. In the lecture we present algebraic aspects of the Hirzebruch signature operator in the category of unitary or Hilbert spaces.

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Idempotent ideals of associative rings

I. V. Kulakivska

Let A be an associative ring with $1 \neq 0$. A two-sided ideal $I \subset A$ is idempotent, if $I^2 = I$. If $J^2 = J$, then $(I+J)^2 = I+J$. Denote by $I(A)$ the set of all idempotent ideals of A. The set $I(A)$ is a commutative band by addition.

If $e \in A$ is an idempotent, then the set AeA idempotent ideal.

We will consider semiperfect rings A satisfying the following two conditions:

(a) Let PrA be the prime radical of A and A/PrA is right Noetherian;

(b) the prime radical PrA is T-nilpotent (right and left).

Theorem. *If a semiperfect ring A satisfies to conditions (a) and (b), then every ideal $I \in I(A)$ has the form $I = AeA$ for some idempotent $e \in A$.*

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Cramer's rule for some two-sided quaternionic matrix equations

I. Kyrchei

Let $\mathrm{M}(n,\mathbb{H})$ be the ring of $n \times n$ quaternion matrices. Denote by $\mathbf{a}_{.j}$ the jth column and by $\mathbf{a}_{i.}$ the ith row of a matrix $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its jth column with the column-vector $\mathbf{b}$, and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its ith row with the row-vector $\mathbf{b}$.

Within the framework of theory of the column and row determinants over the quaternion skew field [1] the following theorem introduces Cramer's rule for two-sided quaternionic matrix equations.

Theorem 1. *Suppose*

$$\mathbf{AXB} = \mathbf{C} \tag{1}$$

is a two-sided matrix equation, where $\{\mathbf{A},\mathbf{B},\mathbf{C}\} \in \mathrm{M}(n,\mathbb{H})$ are known and $\mathbf{X} \in \mathrm{M}(n,\mathbb{H})$ is unknown. If $\mathrm{ddet}\mathbf{A} \neq 0$ *and* $\mathrm{ddet}\mathbf{B} \neq 0$, *then (1) has the unique solution matrix with entries represented as follows*

$$x_{ij} = \frac{\mathrm{rdet}_j(\mathbf{BB}^*)_{j.}\left(\mathbf{c}^{\mathbf{A}}_{i.}\right)}{\mathrm{ddet}\mathbf{A}\,\mathrm{ddet}\mathbf{B}},$$

or

$$x_{ij} = \frac{\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{c}^{\mathbf{B}}_{.j}\right)}{\mathrm{ddet}\mathbf{A}\,\mathrm{ddet}\mathbf{B}},$$

where $\mathbf{c}^{\mathbf{A}}_{i.} := \left(\mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{c}_{.1}), \ldots, \mathrm{cdet}_i(\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{c}_{.n})\right)$ *is the row-vector and*

$$\mathbf{c}^{\mathbf{B}}_{.j} := \left(\mathrm{rdet}_j(\mathbf{BB}^*)_{j.}(\mathbf{c}_{1.}), \ldots, \mathrm{rdet}_j(\mathbf{BB}^*)_{j.}(\mathbf{c}_{n.})\right)^T$$

is the column-vector for all $i = \overline{1,n}$ and $j = \overline{1,n}$.

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Limit theorems for L-functions with an increasing modulus

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Let χ be a Dirichlet character modulo q, and $L(s,\chi)$, $s=\sigma+it$, be a corresponding Dirichlet L–function. The first limit theorem for $L(1,\chi)$ with real character χ, as $q\to\infty$, was obtained by Chowla and Erdős in 1951. In 1971, 1972, Elliott proved limit theorems for $|L(s,\chi)|$ and $\arg L(s,\chi)$ as $q\to\infty$. Stankus [1] generalized the latter results for $L(s,\chi)$.

Let $F(z)$ be a holomorphic normalized Hecke eigen cusp form of weight k for the full modular group. We consider the twisted L-function $L(s,F,\chi)$ defined, for $\sigma>\frac{k+1}{2}$, by the Dirichlet series

$$L(s,F,\chi)=\sum_{m=1}^{\infty}\frac{c(m)\chi(m)}{m^s},$$

where $c(m)$ are the Fourier coefficients of the form $F(z)$. The function $L(s,F,\chi)$ can be analytically continued to an entire function, and has, for $\sigma>\frac{k+1}{2}$, the Euler product expansion over primes.

We suppose that q is a prime number, and, for $Q>2$, denote

$$M_Q=\sum_{q\in Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q),\\ \chi\neq\chi_0}}1,$$

where χ_0 is the principal character mod q. In the report, we consider [2] the weak convergence of probability measures

$$\mu_Q\big(|L(s,F,\chi)|\in A\big),\quad A\in\mathcal{B}(\mathbb{R}),$$

$$\mu_Q\big(\arg L(s,F,\chi)\in A\big),\quad A\in\mathcal{B}(\gamma),$$

and

$$\mu_Q\big(L(s,F,\chi)\in A\big),\quad A\in\mathcal{B}(\mathbb{C}),$$

where

$$\mu_Q(\dots)=M_Q^{-1}\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q)\\ \chi\neq\chi_0\\ \dots}}1,$$

and in place of dots a condition satisfied by a pair $(q,\chi(\mathrm{mod}\ q))$ is to be written. Here $\mathcal{B}(S)$ is the class of Borel sets of the space S, and γ is the unit circle on $\mathbb{C}$.

Similar problems can also be discussed for L-functions of elliptic curves.

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On pro-2-completion of first Grigorchuk group

Yuriy Leonov

Let Gr be the first Grigorchuk group constructed in [1]. It is a well-known 3-generated infinite 2-group, which is subgroup of automorphism group $Aut\ T_2$ of the rooted dyadic tree T_2. Generators of $Gr = \langle a, b, c\rangle$ can be described recursively ($d = bc$) using its action on subtrees:

$$b = (a, c),\ c = (a, d),\ d = (e, b),$$

where a permutes 2 subtrees of level one, e is the neutral element of $Aut\ T_2$.

Let $\overline{Gr}$ be pro-2-completion of Gr. Let's regard the new elements of the group $Aut\ T_2$:

$$z = (q, x)a,\ x = (r, z),\ r = (u, z),\ u = (x, e),\ q = (z, r)$$

and a group $W = \langle z, x, r, u, q\rangle$. Regard also the semigroup W^+ generated by the same set of elements (with positive powers of generators).

Theorem. *1. Pro-2-completion of the group W is a subgroup of $\overline{Gr}$.*
2. Semigroup W^+ is without torsion.

Corollary. *1. Completions of groups Gr and $W' = \langle a, b, c, z\rangle$ coincide.*
2. Groups Gr and W' are not isomorphic.

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On weakly coercive differential polynomials in two variables in the L^∞ norm

D. Limanskii

A differential polynomial $P(D) = \sum_{|\alpha|\le l} a_\alpha D^\alpha$ of order l is called weakly coercive in the isotropic Sobolev space $\overset{\circ}{W}{}^l_\infty(\mathbb{R}^n)$ if it obeys the a priori estimate

$$\sum_{|\alpha|\le l-1} \|D^\alpha f\|_{L^\infty(\mathbb{R}^n)} \le C_1 \|P(D)f\|_{L^\infty(\mathbb{R}^n)} + C_2 \|f\|_{L^\infty(\mathbb{R}^n)},$$

where constants C_1 and C_2 do not depend on $f \in C_0^\infty(\mathbb{R}^n)$.

Denote by $P^l(\xi) := \sum_{|\alpha|=l} a_\alpha \xi^\alpha$ and $P^{l-1}(\xi) := \sum_{|\alpha|=l-1} a_\alpha \xi^\alpha$ the l- and $(l-1)$-homogeneous parts of the polynomial $P(\xi) = \sum_{|\alpha|=l} a_\alpha \xi^\alpha$.

The following result provides an algebraic criterion for weak coercivity in $\overset{\circ}{W}{}^l_\infty(\mathbb{R}^2)$.

Theorem 1. *[1] Let $P(D)$ be a differential polynomial in two variables of order l, and assume that all the coefficients and zeros of $P^l(\xi)$ are real. Then $P(D)$ is weakly coercive in the isotropic Sobolev space $\overset{\circ}{W}{}^l_\infty(\mathbb{R}^2)$ if and only if the polynomials $P^l(\xi)$ and $\operatorname{Im} P^{l-1}(\xi)$ have no common nontrivial real zeros.*

The proof of Theorem 1 is based on the fact that a weakly coercive polynomial in two variables has no multiple real zeros, and on the decomposition of a ratio of two homogeneous polynomials in two variables into a sum of partial fractions.

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On isomorphisms of nilpotent decompositions of finitely generated nilpotent groups of class 3 without torsion

V. Limanskii

A free product $G * H$ of groups G and H in a variety $\mathfrak{M}$ is calculated by formula $G \circ H = G * H/V(G * H)$. Here $V(G * H)$ denotes the verbal subgroup of $G * H$ corresponding to $\mathfrak{M}$. Let $\mathfrak{M}$ be a nilpotent variety of class 3.

Theorem. *Suppose that $G_1 \circ G_2 \circ \cdots \circ G_k = H_1 \circ H_2 \circ \cdots \circ H_l$ are two decompositions of a finitely generated nilpotent group without torsion having indecomposable and non-identity factors. Then $k = l$, and $G_i \cong H_i$, $i \in \{1, \ldots, k\}$, after a suitable renumbering of factors.*

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Differentially finite and endomorphically finite rings

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We discuss some results of our joint work with O. D. Artemovych.

Let R be an associative ring with an identity element. If $\delta(r) = 0$ for almost all $r \in R$, then a derivation $\delta : R \to R$ is called finite. We study rings with every derivations finite and prove

Theorem 1. *Let R be a left Noetherian ring. Then all derivations of R are finite if and only if R is of one of the following types:*

(i) R is a finite ring;

(ii) R is a differentially trivial ring;

(iii) $R = F \oplus S$ is a ring direct sum of a finite ring F and a differentially trivial ring S.

Recall that a ring R having no non-zero derivations is called differentially trivial [1].

We also investigate EF-rings, i.e. rings R with the finite images $Im\sigma$ for any nonidentity ring endomorphism σ of R. Each rigid ring is an EF-ring. Recall that a ring R is called rigid if it has only the trivial ring endomorphisms, i.e., the identity id_R and zero 0_R [2].

Theorem 2. *Let R be a left Artinian ring. Then R is an EF-ring if and only if it is of one of the following types:*

(i) R is a finite ring;

(ii) R is a rigid ring (i.e. R is a rigid field of characteristic 0 or R is isomorphic to some ring of integers modulo a prime power p^n).

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On action of derivations on nilpotent ideals of associative algebras

V. S. Luchko

Let A be an associative algebra (not necessarily finite dimensional) over a field F. Recall that an F-linear mapping $D : A \longrightarrow A$ is called a derivation of A if $D(ab) = D(a)b + aD(b)$ for arbitrary elements $a, b \in A$. Every element $a \in A$ defines an inner derivation $D_a : x \longrightarrow [a, x] = ax - xa$, where $x \in A$. The set of all derivations of an algebra A will be denoted by $Der(A)$. An ideal I of the algebra A is said to be characteristic if $D(I) \subseteq I$ for any derivation $D \in Der(A)$. In [2] G.Letzter has proved that the Levizki radical (the largest locally nilpotent ideal) of an associative ring R is characteristic provided that the additive group R^+ is torsion-free. This result is an analogue for associative rings of a result of B.Hartley [1] about locally nilpotent radical of a Lie algebra in zero characteristic.

If I is a nilpotent ideal of an associative algebra A over a field of positive characteristic then the ideal $I + D(I)$ may be not nilpotent. Thus, it is of interest to find conditions under what the ideal $I + D(I)$ is nilpotent and the sum of all nilpotent ideals is characteristic.

If I is an ideal of an associative algebra A, then the index of nilpotency $n(I)$ is a natural number such that $I^{n(I)} = 0, I^{n(I)-1} \neq 0$. Further, $N(A)$ denotes the sum of all nilpotent ideals of the algebra A. The ideal $N(A)$ is locally nilpotent, i.e. each of its subalgebras generated by finite number of elements from $N(A)$ is nilpotent.

Theorem. *Let A be an associative algebra over a field F, let I be a nilpotent ideal of index n from A and $D \in Der(A)$. Then $I+D(I)$ is a nilpotent ideal of index $\leq n^2$ in the following cases: 1) $charF = 0$; 2) $charF = p > 0$ and $n < p$.*

Corollary. *Let A be an associative algebra over a field F and $N(A)$ be the sum of all nilpotent ideals of the algebra R. Then $N(A)$ is a characteristic ideal of A in the following cases: 1) $charF = 0$; 2) $N(A)$ is a nilpotent ideal of index $< p$, where $p = charF > 0$.*

Note that the restriction on the index of nilpotency in these statements cannot be omitted because there exists an associative algebra A over a field of characteristic p and $D \in Der(A)$ such that $I + D(I)$ is non-nilpotent for a nilpotent ideal I of the algebra A with $n(I) = p$.

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Norm of cyclic subgroups of non-prime order in p-groups ($p \neq 2$)

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In the modern theory of groups an important place is occupied by results, got at the study of groups depending on properties of different characteristic subgroups, in particular, the center of group, the commutant of group and others. Effective direction of researches is also a study of groups in which restrictions are put on the wide class of characteristic subgroups — on Σ-norm. In this abstract the system Σ is the system of all cyclic subgroups of non-prime order. Such Σ-norm is called the norm of cyclic subgroups of non-prime order and is denoted $N_G(C_{\overline{p}})$ (see [1]). Thus, the norm $N_G(C_{\overline{p}})$ is the intersection of normalizers of all cyclic subgroups of non-prime order (on condition that the system of such subgroups is not empty).

The norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order in non-periodic groups was studied in [1]. In the case when the periodic group G coincides with the norm $N_G(C_{\overline{p}})$, all cyclic subgroups of the compound order are invariant in the group G. Non-Dedekind groups of such type were studied in [2] and were called almost Dedekind groups.

Lemma 1. *The norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order of p-group G ($p \neq 2$) is Abelian, if there exist such cyclic subgroup $\langle x\rangle$ of compound order in the group G, that $\langle x\rangle \cap N_G(C_{\overline{p}}) = E$.*

Lemma 2. *If norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order of locally finite p-group G ($p \neq 2$) is non-Abelian, then every cyclic subgroup of compound order of the group G contains commutant of the norm $N_G(C_{\overline{p}})$.*

Corollary 1. *The norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order is Abelian, if the group G contains the subgroup of type (p^2, p^2).*

Theorem 1. *If p-group G ($p \neq 2$) satisfies the minimal condition for subgroups and has non-Abelian norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order, then it is the finite extension of the central quasicyclic subgroup.*

Corollary 2. *The p-group G ($p \neq 2$), which has the non-Abelian norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order, does not contain products of two quasicyclic subgroups.*

Corollary 3. *If the p-group G ($p \neq 2$) satisfies the minimal condition for subgroups and has the non-Abelian norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order, then $[G : N_G(C_{\overline{p}})] \leq \infty$.*

Corollary 4. *If the p-group G ($p \neq 2$) is infinite extension of the norm $N_G(C_{\overline{p}})$, then either the norm $N_G(C_{\overline{p}})$ of cyclic subgroups of non-prime order is Abelian or the group G does not satisfy the minimal condition for subgroups.*

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Algebraic Approach to Problem Solving of Linear Inequalities System

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In the present report the algebraic approach to designing the algorithm of solution of linear inequalities system (LIS) is developed. The essence of the matter is that multisorted algebraic system

(MAS) A_{Constr} in which LIS presented as the expression S is constructively defined. LIS solution is computation of the value as element of A_{Constr} of MAS S - special canonical form of S. The result of applying this approach is the algebraic programs - specifications of MAS. These specifications use notions of inheritance, extensions and morphisms of MAS.
This canonical form permits also to describe algorithms of transformation of multiple integral on polyhedral domain to iterated integral, changing the integration's order, solving the problem of linear programming. We use the idea of projecting the methods of Furier-Motskin and Chernikov. The description of MAS uses:
Sort Variable - linearly ordered set of variables.
Sort ExtCoef - linearly ordered field **Coef**, extended with symbols $-\infty, +\infty$.
Sort LinComb(Coef, Variable) - affine space of linear combinations.
Sort LinUnEqu - algebra of linear inequalities, presented in the form

$$x_n \leq a_1x_1 + \cdots + a_{n-1}x_{n-1}, \quad or \quad x_n \geq a_1x_1 + \cdots + a_{n-1}x_{n-1}$$

Sort VarSegment - sort of elementary segments of solution's space of LIS over **ExtCoef**. Element of **VarSegment** has the form $S = [L(Y), x, G(Y)]$ with semantics $L(Y) \leq x \leq G(Y)$.
Sort Trapezoid - sort of elementary trapezoids in solution's space of LIS over **ExtCoef**. The element is defined by construction

$$T = S_n.S_{n-1}.\ldots.S_1, S_j = [L_j(X_{j-1}), x_j, G_j(X_{j-1}]), X_{k-1} = (x_1, \ldots, x_{k-1})$$

with semantics

$$T = (L_n(X_{n-1}) \leq x_n \leq G_n(X_{n-1}))\& \ldots \&(L_1 \leq x_1 \leq G_1).$$

Sort ConPol - sort of convex polygons in solution's space of LIS over **ExtCoef**. The element can be presented by canonical form

$$P = T_1 + +T_2 + + \ldots + +T_k, T_i \in \textbf{Trapezoid},$$

i.e. splitted on sum of disjointed trapezoids.
Mark in conclusion that this approach is applicable for a wide class of subject domains, it permits to obtain well structured algebraic programs, also to solve problems of synthesis and verification of appropriated algebraic problems.

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On non-periodic groups with non-Dedekind norm of Abelian non-cyclic subgroups

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The intersection of normalizers of all Abelian non-cyclic subgroups of group G is called the norm of Abelian non-cyclic subgroups and is denoted N_G^A. If the norm N_G^A contains Abelian non-cyclic subgroups, then all of them are invariant in it, and subgroup N_G^A is either Dedekind, or $\overline{HA}$-group [1]. Properties of subgroup N_G^A influence on properties of group G, especially, when N_G^A is non-Dedekind.

In abstract authors proceed (see [2]) researches of non-periodic groups with the non-Dedekind norm N_G^A of Abelian non-cyclic subgroups, in particular, in the case when N_G^A is finite extension of infinite cyclic subgroup, which is basic for non-periodic $\overline{HA}$-group.

Theorem 1. *The norm N_G^A of Abelian non-cyclic subgroups of non-periodic group G is Dedekind in each of following cases:*

1) subgroup N_G^A is finite;

2) group G contains Abelian non-cyclic subgroup M, which satisfies the condition $M \cap N_G^A = E$;

3) group G contains the infinite cyclic invariant subgroup $\langle g \rangle$, which satisfies the condition $(\langle g \rangle) \cap N_G^A = E$.

Theorem 2. *Let G be the non-periodic locally soluble group. Suppose that its norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind and is finite extension of infinite cyclic subgroup. If the group G is also finite extension of infinite cyclic subgroup, then its centralizer contains all elements of infinite order of the group and it is the product of invariant cyclic p-group or quaternion group and infinite cyclic group.*

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A_∞-categories via operads and multicategories

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A_∞-algebras and (unital) A_∞-categories are generalizations of dg-algebras and dg-categories. They are related to operads and multicategories in two ways. First of all, operations in (unital) A_∞-algebras and A_∞-categories form a (non-symmetric) dg-operad, which is an instance of enriched multicategory with one object. This dg-operad is a resolution of the corresponding notion for dg-algebras. We discuss various notions of unitality for A_∞-categories and their equivalence in non-filtered case.

Secondly, A_∞-categories form a closed category and, moreover, they are objects of a symmetric closed multicategory. The latter property holds also for unital A_∞-categories. This point of view leads to the notion of a pretriangulated A_∞-category, which is a generalization of a pretriangulated dg-category.

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Spectrum of Prime Fuzzy Subhypermodules

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Let R be a commutative hyperring with identity and M be an unitary R-hypermodule. We introduce and characterize the prime fuzzy subhypermodules of M. We investigate the Zariski topology on FHspec(M), the prime fuzzy spectrum of M, the collection of all prime fuzzy subhypermodules of M.

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On action of outer derivations on nilpotent ideals of Lie algebras

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If L is a Lie algebra over a field K, then a K-linear mapping $D : L \to L$ is called a derivation of L provided that $D([x, y]) = [D(x), y] + [x, D(y)]$ for all elements $x, y \in L$. The set of all derivations of a Lie algebra L will be denoted by $Der(L)$ (it is also a Lie algebra relative to the operation $[D_1, D_2] = D_1D_2 - D_2D_1$). An ideal I of the algebra L is said to be characteristic if $D(I) \subseteq I$ for any derivation $D \in Der(L)$. It is well known that the nilradical (the sum of all nilpotent ideals) of a finite dimensional Lie algebra over a field of characteristic 0 is characteristic. It was shown in [2] that for an arbitrary Lie algebra L (not necessarily finite dimensional) over a field of characteristic 0 and $D \in Der(L)$ the image $D(I)$ of a nilpotent ideal $I \subseteq L$ lies in some nilpotent ideal of the algebra L. The restriction on characteristic of the ground field is essential while proving this assertion.

Using methods that are analogous to ones in [4] during the investigation of behavior of solvable ideals under outer derivations we show that the similar assertion is true for modular Lie algebras.

The main result of the work is the following theorem:

Theorem. *Let I be a nilpotent ideal of nilpotency class n of a Lie algebra L over a field of characteristic 0 or characteristic $p > n + 1$ and $D \in Der(L)$. Then $I + D(I)$ is a nilpotent ideal of the Lie algebra L of nilpotency class at most $n(n + 1)(2n + 1)/6 + 2n$.*

We should note that the estimation of nilpotency class of the ideal $I + D(I)$ from this theorem is rather rough.

Corollary. *Let L be a Lie algebra (not necessarily finite dimensional) over a field F, let $N(L)$ be the sum of all nilpotent ideals of L. If the ideal $N(L)$ is nilpotent, then it is a characteristic ideal in the following cases: a) $char F = 0$; b) $char F = p > 0$ and nilpotency class of $N(L)$ is less than $p - 1$.*

The restriction on nilpotency class in the last statement cannot be omitted. Really, in [1], p.74-75 an example of Lie algebra L of characteristic p is constructed such that the nilradical N of L is of nilpotency class p and $N + D(N)$ is non-solvable for a derivation $D \in Der L$.

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On multiacts over right zero semigroups

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A right act over a semigroup S [1] is the set X with the mapping $X \times S \to X$, $(x, s) \mapsto xs$ such that $x(st) = (xs)t$ for all $x \in X$, $s, t \in S$. Similarly we can define a left act. If two or several semigroups acts on the set X, then we can define a biact and a multiact. A biact [1] over two semigroups S and T is the set X which is a left S-act and a right T-act, and the actions of these semigroups are commutative, i.e., $(sx)t = s(xt)$ for all $s \in S$, $t \in T$, $x \in X$. If the set X is a right act over a family of semigroups $\{S_i | i \in I\}$ and $(xs_i)s_j = (xs_j)s_i$ for all $x \in X$, $s_i \in S_i$, $s_j \in S_j$, then we call X a multiact over the family of semigroups $\{S_i | i \in I\}$. In [2] were described acts over right zero and left zero semigroups. Our results generalize theorems of [2] on multiacts over right zero and left zero semigroups. Namely, the following theorem is proved.

Theorem. *Let X, $\{R_i | i \in I\}$ be the sets, σ_i be the equivalence relations on X for every $i \in I$, and $\sigma_{ij} = \sup\{\sigma_i, \sigma_j\}$ for all $i, j \in I$ and $i \neq j$. For any $r \in R_i$, let Y_r^i be a set of the representatives of the σ_i accordingly, i.e., for all $i \in I$, $r \in R_i$, $x \in X$*

$$|Y_r^i \cap x\sigma_i| = 1.$$

Let the following condition be fulfilled: for every $x \in X$, every $i, j \in I$ such that $i \neq j$, there exists an element $a \in K_{ij}$ (here K_{ij} is the σ_{ij}-class of the element x) such that

$$Y_r^i \cap a\sigma_j = \{a\}.$$

For $r_1, r_2 \in R_i$ we put $r_1 r_2 = r_2$. Then R_i is a right zero semigroup, for all $i \in I$. Put now

$$xr = x' \in Y_r^i \cap x\sigma_i$$

($x \in X$, $r \in R_i$). Then X is a multiact over the family of right zero semigroups $\{R_i | i \in I\}$. Conversely, every multiact over a family of right zero semigroups can be obtained by this way.

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Decomposition numbers for Schur superalgebras $S(m|n)$ for $m+n \leq 4$

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The focus of this talk are Schur superalgebras $S = S(m|n)$ in positive characteristic p. We recall basic structural results in characteristic zero and explain some corresponding results in positive characteristic. In particular, we have computed decomposition numbers and filtrations of costandard modules by simple modules for $S = S(3|1)$, $S(2,2)$, $S(2|1)$ and $S(1|1)$. Knowledge of characters of simple modules for these S allowed us to confirm a conjecture of La Scala - Zubkov about superinvariants of the general linear supergroup $GL(m|n)$.

(Part of the talk is a joint work with A.N. Grishkov and A.N.Zubkov.)

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On certain collections of submodules

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Let R be a ring and let M be a left R-module.

We shall consider the following conditions for a collection $F(M)$ of some submodules of M:

C1. $L \in F(M), L \leq N \leq M \Rightarrow N \in F(M)$;

C2. $L \in F(M), f \in End(M) \Rightarrow (L:f)_M \in F(M)$;

C3. $N, L \in F(M) \Rightarrow N \cap L \in F(M)$;

C4. $N \in F(M), N \in Gen(M), L \leq N \leq M \wedge \forall g \in End(M)_N : (L:g)_M \in F(M) \Rightarrow L \in F(M)$;

C5. $N, K \in F(M), N \in Gen(M) \Rightarrow t_{(K \subseteq M)}(N) \in F(M)$.

Definition. *A non-empty collection $F(M)$ of submodules of a left R-module M satisfying (C1), (C2), (C3) [(C1), (C2), (C3), (C4)] is said to be a preradical [radical] filter of M.*

Let $Rf(M)[\Pr f(M)]$ be the set of all radical [preradical] filters of M.

Theorem. *If M is a semisimple left R-module with a unique homogeneous component then*

(i) $Rf(M) = \Pr f(M)$;

(ii) $(Rf(M), \subseteq)$ is a chain;

(iii) M is a non-zero module of finite composition length then

$$Rf(M) = \{\{\{0\}\}, \{L | L \leq M\}\}.$$

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On quasi-prime differential modules and rings

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Let (R, Δ) be an associative differential ring with nonzero identity, and let (M, D) be a differential module over (R, Δ), where $\Delta = \{\delta_1, \ldots, \delta_n\}$ is the set of pairwise commutative ring derivations, $D = \{d_1, \ldots, d_n\}$ is the set of module derivations consistent with the corresponding ring derivations δ_i.

Recall from [1] that a differential submodule N is *quasi-prime* if there exists an m-system S of R and Sm-system X of M such that N is maximal among differential submodules not meeting X.

A differential submodule N of M is called *differentially prime* if M/N is differentially prime, i. e. if the left annihilator of each of its nonzero submodules coincides with the annihilator of the module M/N.

Theorem 1. *If P is a quasi-prime differential submodule of the differential module M, then P is differentially prime in M.*

We call a differential ring quasi-prime if its zero (differential) ideal is quasi-prime.

Proposition 1. *Let P be a differential ideal of the differential ring R. Then the factor ring R/P is quasi-prime if and only if P is a quasi-prime ideal of R.*

It is clear that any differentially simple ring is quasi-prime. Every differential integral domain is a quasi-prime differential ring. The matrix ring over a differential domain is quasi-prime. It follows from the above proposition, that the factor rings R/M, R/P, where M is maximal among differential ideals, P is prime ideal, are quasi-prime.

A differential ring is quasi-semiprime if it has no nonzero differentially nilpotent ideals.

Theorem 2. *A direct product of quasi-prime differential rings is a quasi-semiprime ring.*

We also consider a question on ultraclosedness of a class of quasi-prime differential modules and rings.

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Homogeneous Markov chains

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Let $P = (p_{ij}) \in M_n(\mathbb{R})$ be a transition matrix for a finite homogeneous Markov chain (see [1, Ch. 6]). In this case $p_{ij} \geq 0$ and $\sum\limits_{j=1}^{n} p_{ij} = 1$ $(i = 1, \dots, n)$, i.e., P is stochastic.

Let $B = (b_{ij}) \in M_n(\mathbb{R})$ be a non-negative matrix. Using B one can construct a simply laced quiver $Q(B)$ in the following way:

(a) the set of vertices $VQ(B)$ of $Q(B)$ is $\{1, 2, \dots, n\}$;

(b) there is one arrow from i to j if and only if $b_{ij} > 0$.

Let $P = \{\alpha_1, \dots, \alpha_n\}$ be a finite poset and $[Q(P)] = B$ the adjacency matrix of $Q(P)$. $B = [Q(P)] = (b_{ij})$ is $(0,1)$-matrix with zero main diagonal. Let $\sum\limits_{j=1}^{n} b_{ij} = \beta_j$ $(j = 1, \dots, n)$ and $D = diag(\beta_1^{-1}, \dots, \beta_n^{-1})$. Then DB is a stochastic matrix with $Q(DB) = Q(P)$. A Markov chain with the transition matrix DB is absorbing.

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An example of the Jordan algebras variety with the almost polynomial growth

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We give here the first example of the variety of Jordan algebras with almost polynomial growth. Necessary definitions see for instance in the books [1] and [2].

Let F be a field of characteristic zero and $\mathbf{V}$ be the variety of linear algebras over F. Let $P_n(\mathbf{V})$ be the vector space of the multilinear polynomials in the first n variables of relatively free algebra. The sequence of dimensions $c_n(\mathbf{V}) = \dim P_n(\mathbf{V})$ defines the growth of the variety $\mathbf{V}$.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be the partition of the number n and χ_λ the corresponding irreducible character of the symmetric group S_n. We can consider $P_n(\mathbf{V})$ as S_n-module and decompose its character $\chi_n(\mathbf{V})$ into the sum of irreducible characters with multiplicities m_λ

$$\chi_n(\mathbf{V}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{1}$$

Let A be the commutative algebra generated by t, a_i, b_i, $i = 1, 2, \ldots$, over F with relations: the product of any words of degree more than one will be zero and also $a_i a_j = 0$, $b_i b_j = 0$, $t^2 = 0$, $a_i b_j = 0$, $tb_i = 0$, $t \ldots a_i a_j = 0$, $t \ldots b_i b_j = 0$, $t \ldots a_i b_j a_{i+1} = -t \ldots a_{i+1} b_j a_i$, $t \ldots b_i a_j b_{i+1} = -t \ldots b_{i+1} a_j b_i$ for any i and j.

Theorem 1. *Let* $\mathbf{V}$ *be the variety generated by the algebra* A. *Then*

1) $\mathbf{V}$ *is the variety of Jordan algebras defined by the identity* $(xy)(zt) \equiv 0$;

2) $\mathbf{V}$ *is the variety of the almost polynomial growth;*

3) $c_n(\mathbf{V}) = k\binom{n}{k+1}$, *where* $k = \left[\frac{n}{2}\right]$ *is the integer part of* $\frac{n}{2}$;

4) *the multiplicities in (1) satisfy the next conditions:* $m_\lambda = 0$ *if* $\lambda = (1^n)$ *or* $\lambda_1 \geq 4$ *or* $\lambda_1 = \lambda_2 = 3$ *and other multiplicities are equal to 1.*

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On a finite group having a normal series with restriction on its factors

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By the Zassenhaus Theorem [1, IV, Satz 2.11] the commutant of a finite group with cyclic Sylow subgroups is a cyclic Hall subgroup such that the corresponding quotient group is also cyclic. Hence the derived length of such group is no greater than 2.

Recall that a group G is bicyclic if it is the product of two cyclic subgroups. The invariants of finite groups with bicyclic Sylow subgroups were found in paper [2]. In particular, it is proved that the derived length of such groups is at most 6 and the nilpotent length of such groups is at most 4.

It is easy to show that a finite group is supersolvable if it has a normal series such that every Sylow subgroup of its factors is cyclic.

In this paper we study a finite group having a normal series with bicyclic Sylow subgroups of its factors. We prove the following

Theorem. *Let* G *be a solvable group having a normal series such that every Sylow subgroup of its factors is bicyclic.*

1) *The nilpotent length of* G *is at most 4 and the derived length of* $G/\Phi(G)$ *is at most 5.*
2) G *contains a normal Ore dispersive subgroup* N *such that* G/N *is supersolvable.*
3) $l_2(G) \leq 2$ *and* $l_p(G) \leq 1$ *for every prime* $p > 2$.
4) G *contains the normal Ore dispersive* $\{2, 3, 7\}'$*-Hall subgroup.*

Here $\Phi(G)$ is the Frattini subgroup of a group G and $l_p(G)$ is the p-length of G.

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Finite state conjugation of linear functions on the ring of n-adic integer numbers

D. Morozov[♭], Yu. Bodnarchuk[♭]

Automorphisms of the one rooted infinite binary tree T_n (the degree of all vertices except the root one equals n+1) can be identified with bijections of the ring Z_n of integer n-adic numbers. For instance, the so-called adding machine ϵ can be defined as the function $x \to x+1, x \in Z_n$. As well known, the centralizer $C_{AutT_n}(\epsilon)$ is a closure of the cyclic group $<\epsilon>$ in the topology of projective limit on $AutT_n$ and consists of the functions $x \to x+p, p \in Z_n,\ C_{AutT_n}(\epsilon) \simeq Z_n^+$.

Here we investigate the conjugation problem for automorphisms of T_n, which can be realized as linear functions of the form $x \to ax+b, a \in Z_n^*, b \in Z_n$ in the finite state automorphisms group $FAutT_n$. Let Z_n^{qp} be the set of quasi-periodic n-adic integers then finite state automorphisms which are linear functions can be characterized in such a manner.

Lemma. *A linear function $f(x) = ax+b$ is a finite state automorphism if and only if $a \in Z_n^{qp*}, b \in Z_n^{qp}$.*

We say that automorphism is a spherical homogeneous if it acts transitively on vertices which are equidistant from the root one. Adding machine ϵ is an example of such an automorphism. As well known all spherical homogeneous automorphisms are conjugate in $AutT_n$.

Theorem. *Two spherical homogeneous finite state automorphisms $f_1(x) = a_1x+b_1, f_2(x) = a_2x+b_2, a_1, a_2, b_1, b_2 \in Z_n^{qp*}$ which are linear functions are conjugate in $FAutT_n$, if and only if $a_1 = a_2$.*

In particular, the spherical homogeneous finite state automorphism in $FAutT_2$ $x \to 5x+1$ is not conjugate to $\epsilon : x \to x+1$. This phenomena can be explained in such a manner. It is easy to prove that if an equation $(x+1)^\chi = 5x+1$ have a solution $\chi \in FAutT_2$ then it has a solution $\chi_0 \in FAutT_2$ such that $0^\chi = 0, 0 \in Z_2$. On the other hand it is known that for spherical homogeneous automorphisms $g_1, g_2 \in AutT_2$ for any pair of points $x, y \in Z_2$ there exists $\chi \in AutT_2$ such that $g_1^\chi = g_2$ and $x^\chi = y$, moreover the last condition defines χ uniquely. Let $\chi_0 = \frac{5^x-1}{4}$ be a function which is defined for a natural x and extended by continuity on Z_2. It is easy to check that $\chi_0 \in AutT_2$, satisfies the equation and $0^{\chi_0} = 0$ so it is unique solution in $AutT_2$ which leaves immovable 0. But $\chi_0 : \frac{1}{3} \to \frac{\sqrt[3]{5}-1}{4}$ and we get that some periodic sequence of digits transforms to the aperiodic one, which implies that $\chi_0 \notin FAutT_2$.

Remark that linear functions $f(x) = 5x+1, f^{-1}(x) = (1/5)(x-1)$, are not conjugated in $FAutT_2$, so $FAutT_2$ is not ambivalent in contrast to $AutT_2$.

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On a Heyde characterization theorem for discrete Abelian groups

M. Myronyuk

A lot of studies were devoted to characterizations of a Gaussian distribution on the real line. Specifically, in 1970 Heyde proved the following theorem.

Heyde theorem ([1, §13.4.1]). *Let ξ_j, $j = 1, 2, ..., n$, $n \geq 2$, be independent random variables. Let α_j, β_j be nonzero constants such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric then all random variables ξ_j are Gaussian.*

Let X be a locally compact separable Abelian metric group, $\mathrm{Aut}(X)$ the set of topological automorphisms of X. Let ξ_j, $j = 1, 2, ..., n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. Consider the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Formulate the following problem.

Problem 1. Describe groups X for which the symmetry of the conditional distribution of the linear form L_2 given L_1 implies that all distributions μ_j are either Gaussian or belong to a class of distributions that can be considered as a natural analogue of the class of Gaussian distributions.

Problem 1 has not been solved, nevertheless it was studied in different important subclasses of the class of locally compact Abelian groups. In [2] Problem 1 was completely solved in the class of finite Abelian groups, and then in [3] it was solved in the class of countable discrete Abelian groups.

Formulate now the following general problem.

Problem 2. Let X be a locally compact separable Abelian metric group. Assume that the conditional distribution of the linear form L_2 given L_1 is symmetric. Describe possible distributions μ_j.

Problem 2 was solved in the class of finite Abelian groups in [4]. We solve Problem 2 in the class of countable discrete Abelian groups.

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About Sets of Congruences on Some Subsemigroups of the Semigroup of Linear Relations

M. I. Naumik

Let V be a left-vector space over a substance. We recall that a linear relation on V is a subspace of a space $V \bigoplus V$ and denote the multiplicative semigroup of all linear relations on V by $LR(V)$.

Let $LV(V)$ and $LW(V)$ be subsemigroups of the semigroup $LR(V)$, and consequently of all partial coisolated linear transformations and all partial linear transformations. The description of congruences on these semigroups one can find in [1, 2].

Theorem 1. *A set of congruences on $LW(V)$ is a semiset of the set of all binary relations on $LW(V)$. In particular, it is a distributive set.*

Theorem 2. *A set of congruences on $LV(V)$ is a semiset of the set of all binary relations on $LV(V)$. In particular, it is a distributive set.*

These results are the continuations of the development of the idea by A.I. Maltsev [1].

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On nondegeneracy of Tate pairing for curves over complete discrete valuation fields with pseudofinite residue fields

V. Nesteruk

We study the Tate pairing for curves over complete discrete valuation fields with pseudofinite residue fields.

Let C be an absolutely irreducible projective curve defined over a field k and let K be an algebraic extension of k, $(m, \mathrm{char}(k)) = 1$. For divisor classes $x \in \mathrm{Cl}^0(K(C))[m]$ and $y \in \mathrm{Cl}^0(K(C))/m\mathrm{Cl}^0(K(C))$ there are coprime divisors D and R such that $x = [D]$ and $y = [R] + m\mathrm{Cl}^0(K(C))$, and there exists a function $f \in K(C)$ such that $(f) = mD$.

Definition. *The Tate pairing*

$$t_m \colon \mathrm{Cl}^0(K(C))[m] \times \mathrm{Cl}^0(K(C))/m\mathrm{Cl}^0(K(C)) \longrightarrow K^*/(K^*)^m$$

is defined by $t_m(x, y) = f(R)$.

F. Hess in [1] presented an elementary proof of the nondegeneracy of the Tate pairing for curves over finite fields. Also, M. Papikian in [2] proved the nondegeneracy of the Tate pairing for curves over complete discrete valuation fields with finite residue field. We prove the nondegeneracy of the Tate pairing for curves over complete discrete valuation fields with pseudofinite residue field. Namely,

Theorem. *The Tate pairing t_m is nondegenerate for curves over complete discrete valuation fields with pseudofinite residue field.*

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Composition Nominative Algebras as Computer Program Formalism

Mykola Nikitchenko

The aim of the talk is to present the definition, properties and applications of *Composition Nominative Algebras* (CNA), which can be considered as adequate models of program semantics of various levels of abstraction and generality.

Formalization of program semantics is based on the following main principles.

Development Principle (from abstract to concrete): the notion of program should be introduced as a process of its development which starts from abstract understanding capturing essential program properties and proceeds to more and more concrete considerations, thus gradually revealing the notion of program in its richness.

Compositionality Principle: programs can be considered as functions which map input data into results, and which are constructed from simpler programs (functions) with the help of special operations, called compositions.

Nominativity Principle: structures of programs data are based on nominative (naming) relations.

In accordance with these principles, CNA can be defined as algebras of functions over nominative data with compositions as operations.

Three types of CNA are investigated, which correspond to *abstract, collective and hierarchic levels* in data considerations. On the collective level such data types as *presets, sets, and nominates* are specified. Presets are considered as collections without equality of their elements, sets are presets with equality and membership relation, and nominates are specified as collections of named elements. The main attention is paid to the hierarchical level of data construction. Algebras of this level are very expressive and can be used to construct formal models of specification, programming and database languages.

The main problems studied for CNA are various completeness problems, especially *problems of compositional and computational completeness*. Special notions of abstract computability over various data structures – natural computability of functions and determinant computability of compositions – are introduced; complete classes of computable functions and compositions of various abstraction levels are defined. For example, in the simple case of functions over nominates a finite set of names V and arbitrary preset of basic values W is built on, the complete class of corresponding computable functions precisely coincides with the class of functions obtained by closure of naming, denaming and checking functions under multiplication, iteration and overlaying compositions.

CNA, which were primarily oriented on programming, can be also considered in a more general setting as formal models of predicate languages. Such considerations lead to special kinds of logics called *Composition Nominative Logics*. Logics of various abstraction levels are defined, corresponding calculi are constructed, their *soundness and completeness/incompleteness proved* (details are presented in [1]).

In the whole, we can say that CNA can be considered as simple but powerful formal models of program semantics which are also applicable to some other important problem domains, in particular, to mathematical logic and computability theory.

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Groupoid of idempotents of a finite semigroup

Boris V. Novikov

Let S be a finite semigroup, $E(S)$ the set of its idempotents. Define on $E(S)$ a new operation $*$: if $e, f \in E(S)$ then $e * f$ is the idempotent of the monogenic semigroup $\langle ef \rangle$. Denote the obtained groupoid by E^*. It is idempotent since $e * e = e$ for any $e \in E^*$.

In some cases E^* turns out to be a semigroup:

Proposition 1. *Let $S = M(G; I, \Lambda; P)$ be a completely simple (finite) semigroup. Then E^* is a rectangular $(I \times \Lambda)$-band.*

For completely 0-simple semigroups the situation is more complicated:

Proposition 2. *Let $S = M^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup, $e_{i\lambda}$, $e_{j\mu}$, $e_{k\nu}$ $(i, j, k \in I,\ \lambda, \mu, \nu \in \Lambda)$ its nonzero idempotents. Then*

$$(e_{i\lambda} * e_{j\mu}) * e_{k\nu} = \begin{cases} e_{i\nu}, & \text{if } p_{\lambda j}, p_{\mu i}, p_{\mu k}, p_{\nu i} \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$e_{i\lambda} * (e_{j\mu} * e_{k\nu}) = \begin{cases} e_{i\nu}, & \text{if } p_{\mu k}, p_{\nu j}, p_{\lambda j}, p_{\nu i} \neq 0, \\ 0 & \text{in opposite case.} \end{cases}$$

Thus E^* is not a semigroup generally speaking. Recall that a groupoid is **diassociative** if each its 2-generated subgroupoid is associative.

Theorem 1. *For any finite semigroup S the groupoid E^* is diassociative.*

Corollary 1. *Let E^* contain the zero and $e, f \in E^*$. Then*

$$e * f = 0 \Longrightarrow f * e = 0.$$

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On groups with separating subgroups relative systems of indecomposable cyclic subgroups

Oksana Odintsova

A subgroup S of group G is called a separating subgroup if all subgroups from some system Σ of subgroups of the group G are normal in G when they don't belong to S. And such group G is called a group with separating subgroups with regard to system Σ. An intersection M of all separating subgroups of G is called a separator of group G. The separator M coincides with a subgroup of G which is generated by all non-normal subgroups of system Σ [1].

A group with separating subgroup relative systems of proper subgroups of this group is called $H(S)$-group. The description of the $H(S)$-groups is given in theorem 1.4.2 [2].

As is generally known a class of groups with normal cyclic subgroups, in particular with normal indecomposable cyclic subgroups, coincides with a class of Dedekind groups (the groups with normal proper subgroups).

When we consider the groups with different separating subgroups relative systems of subgroups it turns out that a class of groups with separating subgroups relative systems of cyclic subgroups coincides with a class of the $H(S)$-groups. But a class of groups with separating subgroups relative systems of indecomposable cyclic subgroups appears to be wider than the class of the $H(S)$-groups. Expansion happened due to the groups of type 2 of the theorem 2.

Theorem 1. *Each group G (which is not H(S)-group) with separating subgroups relative systems of indecomposable cyclic subgroups is periodic.*

Theorem 2. *A group G is non-Dedekind group with separating subgroups relative systems of indecomposable cyclic subgroups if and only if G is one of following types:*

1. *G is a direct product of proper Sylow p_i-subgroups P_i, $i \in I$, each of them contains subgroup S_i (S_i is normal in the group G) and all subgroups of P_i are normal in G, when they do not belong to S_i (P_i is H(S)-group with separator S_i);*
2. *G=$A\times B$, when A and B are Hall subgroups of group G, subgroup A is Dedekind group or a group of type 1 of this theorem and subgroup B cannot be Dedekind group or a group type 1 of this theorem.*

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Actions of inverse semigroups on rooted trees

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Let T_n, $n \geq 2$, be a rooted n-regular tree. Denote by $ISA(T_n)$ the inverse semigroup of partially defined automorphisms of T_n. An inverse semigroup S is said to act level transitively by partial automorphisms on the rooted tree T_n if there exists an inverse semigroup homomorphism from S onto some level transitive inverse subsemigroup of $ISA(T_n)$.

Recall, that the canonical partial order "$\leq$" on an inverse semigroup S is defined by the rule

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S),$$

where $E(S)$ denotes the set of idempotents of S. For any nonempty subset A of S the set

$$A\omega = \{b \in S : a \leq b \text{ for some } a \in A\}$$

is called the closure of A. If $A\omega = A$ then A will be called closed.

Let H be some closed inverse subsemigroup of an inverse semigroup S and an element $a \in S$ such that $aa^{-1} \in H$. Then a set $(Ha)\omega$ is called a right coset of S by H. The different cosets are disjoint. One of the cosets is H itself. The cardinality of the set of all right cosets of S by H is called the index of H in S and denoted by $[S : H]$. The notions of closed subset and right coset were introduced in [1].

Theorem. *Let S be an inverse semigroup. If there exists a descending chain*

$$S = S_0 \supset S_1 \supset S_2 \supset \ldots$$

of closed inverse subsemigroups such that

$$\bigcap_{i=0}^{\infty} S_i = E(S)$$

and all the indexes $[S_i : S_{i+1}]$, $i \geq 0$, are equal to some $n \geq 2$ then the inverse semigroup S acts level transitively by partial automorphisms on the rooted tree T_n.

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Doubling of metric spaces

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We shall use standard definitions from metric geometry([1]). Let (X, ρ) be a metric space, (X', ρ') be an isometric copy of (X, ρ) and f be some isometry from X to X'. Fix a positive number $a \in R^+$ and define a non-negative symmetric real function d_f on the disjoint union $X \sqcup X'$ by the rules:

- if $x = y$ then $d_f(x, y) = 0$

- if $x,\ y\ \in X$ then $d_f(x,y) = \rho(x,y)$;
- if $x,\ y\ \in X'$ then $d_f(x,y) = \rho'(x,y)$;
- if $x\ \in X,\ y\ \in X'$ and $y = f(x)$ then $d_f(x,y) = a$;
- if $x\ \in X,\ y\ \in X'$ and $y \neq f(x)$ then $d_f(x,y) = a + \rho'(f(x),y)$.

The function d_f is a metric on the set $X \sqcup X^{'}$. We call $(X \sqcup X^{'}, d_f)$ a *doubling of metric space X* and denoted $2X(f,a)$.

Proposition 1. *Let f_1 and f_2 be some isometries of (X,ρ). Then spaces $2X(f_1,a)$ and $2X(f_2,a)$ are isometric.*

We say that a metric space (X,ρ) is *a Cayley's space* if there exists some group H and a system of generators S of this group such that the Cayley graph (H,S) is isometric to (X,ρ) as a metric space (see [2]).

Theorem 1. *If (X,ρ) is a Cayley's metric space, then $2X(f,1)$ is a Cayley's metric space.*

Theorem 2. *Let (X,ρ) be a metric space such that there exists a positive number a such that $a > diamX$. Then the isometry group of doubling of metric space X is isomorphic as a permutation group to the direct product of isometry group of spaces X and the symmetric group S_2:*

$$Is2X(f,a) = IsX \times S_2. \tag{1}$$

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On the Monoids over which all quasi-filters are trivial

R. Oliynyk♭, M. Komarnytskyi♭

Let S be a semigroup with 0 and 1. The terminology and necessary definitions can be found in [1]. Each left $S-$act A is assumed to be unitary (i. e., $1A = A$) and centered (i.e., $0a = s0 = 0$ where $0 \in A$ for all $s \in S$ and $a \in A$).

Let A and B be two left $S-$acts. Recall that the mapping $f : A \to B$ is called homomorphism if $f(sa) = sf(a)$ for all $s \in S$ and $a \in A$. We consider category of left $S-$acts and their homomorphisms and denote it by $S - Act$.

A quasi-filter (see [2]) of S is defined to be subset $\mathcal{E}$ of $Con(S)$ satisfying the following conditions:

1. If $\rho \in \mathcal{E}$ and $\rho \subseteq \tau \in Con(S)$, then $\tau \in \mathcal{E}$.
2. $\rho \in \mathcal{E}$ implies $(\rho : s) \in \mathcal{E}$ for every $s \in S$.
3. If $\rho \in \mathcal{E}$ and $\tau \in Con(S)$ such that $(\tau : s), (\tau : t)$ are in $\mathcal{E}$ for every $(s,t) \in \rho \backslash \tau$, then $\tau \in \mathcal{E}$.

For example, if σ is some Rees congruence on S then the family

$$\mathcal{E}_\sigma = \{\tau \mid \quad \tau \in Con(S), \tau \vee \sigma = 1_S\}$$

is quasi-filter. We call it $\sigma-$ quasi-filter.Also we call a quasi-filter $\mathcal{E}$ trivial if either it contains $\triangle_S$ or only contains $\bigtriangledown_S$, when $\triangle_S = \{(s,s)|s \in S\}$ and $\bigtriangledown_S = \{(s,t)|s,t \in S\}$.

A monoid S is called perfect (see [3]) if every left $S-$act has a projective cover.

The next theorem describes monoids which have only trivial quasi-filters and is an analog of well known result for modules (see [4]).

Theorem. *Let S be commutative monoid. Then the following statements hold:*

1. All quasi-filters $\mathcal{E}$ are trivial.

2. All σ-quasi-filters $\mathcal{E}_\sigma$ are trivial.

3. S is a perfect monoid.

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On number of generators of ideals of prime $SPSD$-rings

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Let A be an associative ring with $1 \neq 0$. Denote by $\mu_r^* = \max\limits_{I \subseteq A} \mu_r(I)$, where $\mu_r(I)$ is the minimal number of generators of a right ideal I of A. Analogously, one can define $\mu_l^*(A)$. By definition, A is principal right ideal ring.

We write $SPSD$-ring A for a semiperfect semidistributive ring A. Following [1, Ch. 14] we say that a ring A is a tiled order if A is a prime right Noetherian $SPSD$-ring with the nonzero Jacobson radical.

Proposition 1. *Let $I \subseteq A$ be a two-sided ideal of a tiled order A. Then $\mu_r(I) = \mu_l(I)$.*

Denote by $\mu_t^*(A) = \max \mu_r(I)$, where I is a two-sided ideal of A.

Let s be a number of vertices of the quiver $Q(A)$ of a tiled order A.

Theorem 1. *The following equalities for a reduced tiled order A hold:*

$$\mu_r^*(A) = \mu_l^*(A) = \mu_t*(A) = s.$$

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On asymptotic properties of modular Lie algebras

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Let $L(X)$ be a free Lie algebra of finite rank over a field of positive characteristic. Let G be a finite nontrivial group of homogeneous automorphisms of the algebra $L(X)$. It is known that the subalgebra of invariants $H = L^G$ is infinitely generated [2]. Our goal is to determine how big is its free generating set. Let $Y = \cup_{n=1}^{\infty} Y_n$ be the free homogeneous generating set of H, where the elements Y_n have degree n with respect to X. In case of characteristic zero there is an exact formula for the generating function $\mathcal{H}(Y,t) = \sum_{n=1}^{\infty} |Y_n| t^n$ [3]. In case of a field of positive characteristic we describe the growth of the generating function and prove that the sequence $|Y_n|$ grows exponentially. Our arguments rely on [1], [3].

Also we consider the subalgebra growth sequence. Let L be a finitely generated restricted Lie algebra over a finite field F. By $a_n(L)$ we denote the number of restricted subalgebras $H \subset L$ such that $\dim_K L/H = n$ for all $n \geq 0$. We obtain the *subalgebra growth* sequence $\{a_n(L) | n \geq 0\}$. This notion is similar to the subgroup growth in group theory and was considered in [6], [4], [5]. We find an upper bound on the subalgebra growth for a finitely generated metabelian restricted Lie algebra over a finite field.

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Matrix polynomial equations and its solutions

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Let F be an algebraically closed field of characteristic zero

$$X^m + A_1 X^{m-1} + \cdots + A_{m-1} X + A_m = 0 \tag{1}$$

be matrix equation, where A_i are $n \times n$ matrices over F, $i = 1, \ldots, m$, X be a variable matrix. The polynomial matrix

$$A(\lambda) = I\lambda^m + A_1 \lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m,$$

where I is identity matrix, $\lambda \in F$ will be called the characteristic matrix of the matrix equation (1). The roots of polynomial $\Delta(x) = \det A(\lambda)$ are characteristic roots of equation (1).

The solvability of the matrix equation (1) depends on the multiplicities of its characteristic roots. If the characteristic roots of equation (1) have multiplicities 1, this equation has k solutions, where $m^n \leqslant k \leqslant \binom{mn}{n}$ [1]. It follows from [2], that the equation (1) is solvable whenever its characteristic roots have multiplicities less or equal 2. In this case we investigate the number of its solutions.

Theorem 1. *Suppose that the characteristic roots of the matrix equation (1) have multiplicities 2 and the elementary divisors of its characteristic matrix $A(\lambda)$ are pairwise relatively prime. Then equation (1) is solvable and the number of its solutions k satisfies the condition*

$$1 \leqslant k \leqslant \sum_{i=0}^{n-t} \binom{n-i}{i} \binom{r}{n-i},$$

where r is the number of the different characteristic roots of equation (1),

$$t = \begin{cases} q, & \text{if } \quad n = 2q \\ q+1, & \text{if } \ n = 2q+1. \end{cases}$$

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On the stable range of rings of matrices

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It is important to study the relationship between the stable range of the ring $M(n, R)$ of matrices of order n over ring R and the stable range of the ring R itself. It is known [1,2] that if the stable range of R is equal to 1 or 2, then the stable range of $M(n, R)$ also equals 1 or 2 respectively. We present the class of matrices of stable range 1 over R such that R can have stable rang greater than 1.

Let R be an adequate ring, i.e. R be domain of integrity in which every finitely generated ideal is principal and for every $a, b \in R$ with $a \neq 0$, a can be represented as $a = cd$, where $(c, b) = 1$ and $(d_i, b) \neq 1$ for any non-unit factor d_i of d [3]. The collection of matrices $(A_1, \ldots, A_k)$, $A_i \in M(n, R)$, $i = 1, \ldots, k$ is called primitive if $A_1V_1 + \cdots + A_kV_k = I$, for some matrices $V_i \in M(n, R)$, $i = 1, \ldots, k$, where I is the identity matrix.

Theorem 1. *Let R be an adequate ring, $M'(2, R)$ the set of matrices $A = \|a_{ij}\|_1^2$, $a_{ij} \in R$ such that $(a_{11}, a_{12}, a_{21}, a_{22}) = 1$. The stable range of the set of matrices $M'(2, R)$ equals one, in other words for every primitive pair of matrices (A, B), $A, B \in M'(2, R)$ there exists the matrix $P \in M(2, R)$ such that $AP + B = Q$, where Q is invertible matrix in $GL(2, R)$.*

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On 0-homology of categorical at zero semigroups

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The 0-cohomology and 0-homology of semigroups were introduced in [1] and [2] as a generalizations of Eilenberg-MacLane cohomology and homology. The one of possible applications of the 0-cohomology and 0-homology is the computation of classical cohomology and homology of semigroups.

It was shown in [2] that the first 0-homology group of a semigroup with zero S is isomorphic to the first homology group of semigroup $\overline{S}$, which is called 0-reflector of S. The 0-homology groups of S of greater dimensions in the general case are not isomorphic to the homology groups of $\overline{S}$.

We show that for the categorical at zero semigroups such an isomorphism can be built for all dimensions.

Definition. *A semigroup S is called categorical at zero if $xyz = 0$ implies $xy = 0$ or $yz = 0$.*

Theorem. *If S is categorical at zero then the 0-homology group $H_n^0(S, A)$ is isomorphic to the homology group $H_n(\bar{S}, A)$ for all $n \geq 0$ and every module A, which is considered as a 0-module over S in the first case and as a module over $\overline{S}$ in the second case.*

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On tensor product of locally compact modules

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Let R be a topological ring with identity, and let ${}_R\mathfrak{L}$ $\left(\text{respectively}, \mathfrak{L}_R\right)$ be the category of left (respectively, right) unital locally compact modules over R. For a locally compact module X, we denote by $c(X)$ the connected component of X and by $k(X)$ the submodule of compact elements of X.

Definition. *A topological module X is said to be compactly generated if it admits a compact set of topological generators.*

Theorem 1. *Let G be an abelian topological group. For any $X \in \mathfrak{L}_R$ and $Y \in {}_R\mathfrak{L}$, let $B(X \times Y, G)$ be the group of all hypocontinuous R-balanced mappings from $X \times Y$ to G, endowed with the compact-open topology.*

(i) If X, Y are compactly generated and G is compact without small subgroups, then $B(X \times Y, G)$ is locally compact.

(ii) If X, Y are compact and G is without small subgroups, then $B(X \times Y, G)$ is discrete.

(iii) If X, Y are discrete and G is compact, then $B(X \times Y, G)$ is compact.

Corollary 1. *Let $X \in \mathfrak{L}_R$ and $Y \in {}_R\mathfrak{L}$. If X, Y are compactly generated, compact, or discrete, then $X \otimes_R Y$ is locally compact, compact or discrete, respectively.*

Theorem 2. *Let $X \in \mathfrak{L}_R$ and $Y \in {}_R\mathfrak{L}$.*

(i) *If $Y = k(Y)$, then $X \otimes_R Y \cong X/c(X) \otimes_R Y$.*

(ii) *If $Y = c(Y)$, then $X \otimes_R Y \cong X/k(X) \otimes_R Y$.*

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On semiscalar equivalence of polynomial matrices

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Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. By $M_{m,n}(\mathbb{F})$ we denote the set of m-by-n matrices over $\mathbb{F}$ and by $M_{m,n}(\mathbb{F}[x])$ the set of m-by-n matrices over the polynomial ring $\mathbb{F}[x]$. For any matrix $C(x) \in M_{n,n}(\mathbb{F}[x])$, let $C^*(x)$ denote the adjoint matrix of $C(x)$, i.e., $C^*(x)C(x) = C(x)C^*(x) = I_n \det C(x)$, where I_n is the $n \times n$ identity matrix.

For $b(x) = (x-\beta_1)^{k_1}(x-\beta_2)^{k_2}\dots(x-\beta_r)^{k_r} \in \mathbb{F}[x]$ and $A(x) \in M_{m,n}(\mathbb{F}[x])$ we define the matrix $M[A,b] = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_r \end{bmatrix} \in M_{mk,n}(\mathbb{F})$, where $N_j = \begin{bmatrix} A(\beta_j) \\ A^{(1)}(\beta_j) \\ \vdots \\ A^{(k_j-1)}(\beta_j) \end{bmatrix} \in M_{mk_j,n}(\mathbb{F})$, $j = 1, 2, \dots, r$, and $k = k_1 + k_2 + \dots + k_r$. The Kronecker product of matrices $C(x)$ and $D(x)$ is denoted by $C(x) \otimes D(x)$.

Matrices $A(x), B(x) \in M_{n,n}(\mathbb{F}[x])$ are said to be *semiscalar equivalent* [1] if there exist matrices $P \in GL(n, \mathbb{F})$ and $Q(x) \in GL(n, \mathbb{F}[x])$ such that

$$A(x) = PB(x)Q(x).$$

We give a criterion of semiscalar equivalence of nonsingular polynomial matrices over an algebraically closed field of characteristic zero.

Theorem 1. *Let nonsingular matrices $A(x), B(x) \in M_{n,n}(\mathbb{F}[x])$ be equivalent and $S(x) = \operatorname{diag}((s_1(x), s_2(x), \ldots, s_n(x))$ be their Smith normal form. For $A(x)$ and $B(x)$ define the matrix $D(x) = \Big(\Big(s_1(x)s_2(x)\cdots s_{n-1}(x)\Big)^{-1} B^*(x)\Big) \otimes A^T(x) \in M_{n^2,n^2}(\mathbb{F}[x])$.*

The matrices $A(x)$ and $B(x)$ are semiscalar equivalent if and only if there exists a vector $t = [t_1, t_2, \ldots, t_{n^2}]^T$ over $\mathbb{F}$ such that $M[D, s_n]t = 0$ and the matrix

$$\begin{bmatrix} t_1 & t_2 & \ldots & t_n \\ t_{n+1} & t_{n+2} & \ldots & t_{2n} \\ \hdotsfor{4} \\ t_{n^2-n+1} & t_{n^2-n+2} & \ldots & t_{n^2} \end{bmatrix}$$

is nonsingular.

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On finite simple paramedial quasigroups

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The problem "Classify the finite simple paramedial quasigroups" is proposed by Jaroslav Jezek and Tomas Kepka at Loops '03, Prague 2003 (http://en.wikipedia.org/wiki/Problems-in-loop-theory-and-quasigroup-theory).

A quasigroup ([1]) $(Q, \cdot)$ is called a T-quasigroup, if there exist an abelian group $(Q, +)$, its automorphisms φ, ψ and a fixed element g such that $x \cdot y = \varphi x + \psi y + g$ for all $x, y \in Q$ [2]. A quasigroup $(Q, \cdot)$ with identity $xy \cdot uv = xu \cdot yv$ is medial. A T-quasigroup is medial iff $\varphi\psi = \psi\varphi$ [1]. A quasigroup $(Q, \cdot)$ with identity $xy \cdot uv = vy \cdot ux$ is paramedial [2]. A T-quasigroup is paramedial iff $\varphi^2 = \psi^2$ [2].

Simple medial quasigroups are described in [3]. Finite simple T-quasigroups are researched in [4]. A quasigroup $(Q, \cdot)$ is α-simple, if this quasigroup does not contain a non-trivial congruence that is invariant relative to a permutation α of the set Q.

Theorem 1. *A finite paramedial quasigroup $(Q, \cdot)$ with the form $x \cdot y = \varphi x + \psi y + g$ over an abelian group $(Q, +)$ is simple if and only if: (i) $(Q, +) \cong \bigoplus_{i=1}^{n} (Z_p)_i$; (ii) the group $\langle \varphi, \psi \rangle$ is an irreducible subgroup of the group $GL(n, p)$ in case $n > 1$, the group $\langle \varphi, \psi \rangle$ is any subgroup of the group $Aut(Z_p, +)$ in case $n = 1$.*

The quasigroup $(Q, \cdot)$ in case $|\,Q\,| > 1$ can be quasigroup from one of the following disjoint quasigroup classes:

1. *$x \cdot y = \alpha\varphi_1 x + \alpha\psi_1 y + g$, where $x \circ y = \varphi_1 x + \psi_1 y$ is a paramedial α-simple distributive quasigroup, $\alpha, \varphi_1, \psi_1 \in Aut(Q,+)$, $\varphi_1 + \psi_1 = \varepsilon$, $\varphi_1 + \varphi_1 = \varepsilon$, $\varphi_1\alpha\varphi_1 = \psi_1\alpha\psi_1$;*

2. *$\psi = -\varphi$; $(Q,\cdot)$ is a medial unipotent quasigroup, quasigroup $(Q,\cdot)$ is isomorphic to quasigroup $(Q,\circ)$ with the form $x \circ y = \varphi x - \varphi y$ over the group $(Q,+)$.*

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Network weight in paratopological groups

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A paratopological group is a pair (G, τ) consisting of a group G and a topology τ on G making the group operation $m\colon G \times G \to G$ on G continuous.

A family $\mathcal{N}$ of subsets of a topological space X is a network for X if for every point $x \in X$ and any neighborhood U of x there exists an $M \in \mathcal{N}$ such that $x \in M \subset U$. The network weight $nw(X)$ of a space X is the smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network for X.

Theorem 1. *Let G be a paratopological group, which is algebraically generated by its symmetric subspace X. Then $nw(X) = nw(G)$.*

Corollary 1. *Let G and H be paratopological groups, $G * H$ — their free product (see [2]). Then $nw(G * H) = max\{nw(G), nw(H)\}$.*

For every topological space there exist free paratopological group $F_p(X)$ on X and free abelian paratopological group $A_p(X)$ on X (see [1]).

Corollary 2. *Let X be a T_1-space. Then $nw(F_p(X)) = nw(A_p(X)) = |X|$.*

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A necessary and sufficient condition for a table algebra to originate from an association scheme

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A table algebra is a C-algebra with nonnegative structure constants was introduced in [1]. As a folklore example, the adjacency algebra of an association scheme (or homogeneous coherent configuration) is an integral table algebra. On the other hand, the adjacency algebra of an association scheme has a special character which is called the *standard character*, see [2]. We generalize the concept of standard character from adjacency algebras to C-algebras. This generalization enables us to find a necessary and sufficient condition for a commutative table algebra to originate from an association scheme.

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On the growth of Poisson PI algebras

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Let K be arbitrary field. A vector space A is called a *Poisson algebra* provided that, beside addition, it has two K-bilinear operations which are related by derivation. First, with respect to multiplication, A is a commutative associative algebra with unit; denote the multiplication by $a \cdot b$ (or ab), where $a, b \in A$. Second, A is a Lie algebra; traditionally here the Lie operation is denoted by the Poisson brackets $\{a, b\}$, where $a, b \in A$. It is also assumed that these two operations are connected by the Leibnitz rule

$$\{a \cdot b, c\} = a \cdot \{b, c\} + b \cdot \{a, c\}, \qquad a, b, c \in A.$$

Let V be a variety of Poisson algebras, $F(V)$ be a countable rank relatively free algebra of the variety V and $P_n(V) \subset F(V)$ be the subspace of all the multilinear elements of degree n in $\{x_1, \ldots, x_n\}$.

Theorem 1. *Let V be a variety of Poisson algebras over arbitrary field, satisfying the identities*

$$\{\{x_1, y_1\}, \{x_2, y_2\}, ..., \{x_m, y_m\}\} = 0,$$

$$\{x_1, y_1\} \cdot \{x_2, y_2\} \cdot ... \cdot \{x_m, y_m\} = 0,$$

for some m. Then there exist numbers N, α, β and such integer $d \in \{1, 2, ..., s\}$ that

$$n^\alpha d^n \leq dim\ P_n(V) \leq n^\beta d^n$$

for any $n \geq N$.

Corollary 1. *If the characteristic of arbitrary fields is not equal to two then there exists no variety of Poisson algebras with intermediate growth between polynomial and exponential.*

Theorem 2. *In the case of characteristic zero of the base field, a variety of Poisson algebras V has polynomial growth if and only if there exists a constant C such that $m_\lambda = 0$ in the sum*

$$\chi_n(V) = \chi(P_n(V)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

whenever $n - \lambda_1 > C$.

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On semigroups and similar groupoids

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Let X be a set, let "$\star$" and "$\times$" be two operations on X such that the Cayley tables of these operations may be different only for the diagonal elements. Then we say that the groupoids $(X, \star)$ and $(X, \times)$ are *similar*. We also say that "$\star$" and "$\times$" are *similar operations*. If an element x is such that $x \star x = x \times x$ then we say that x is a *θ-element*.

The following question is of our interest: *if two groupoids are similar and one of them is a semigroup, under which conditions does the other one become a semigroup?*

Let $(X, \star)$ be a semigroup and $(X, \times)$ be a groupoid similar to $(X, \star)$. We say that *condition A* holds if $(u \times u) \times z = u \times (u \times z)$, $(z \times u) \times u = z \times (u \times u)$ for all elements u which are not θ-elements, and for all elements z. We say that *condition B* holds if $(x \times y) \times u = x \times (y \times u)$, $(u \times x) \times y = u \times (x \times y)$ for all elements u which are not θ-elements, and for all elements x, y such that $u = x \times y$.

Lemma 1. *Let $(X, \star)$ and $(X, \times)$ be similar groupoids, and $(X, \star)$ be a semigroup. Then $(X, \times)$ is a semigroup if and only if both conditions A and B are fulfilled.*

Theorem. *Let $(X, \star)$ and $(X, \times)$ be similar groupoids, and $(X, \star)$ be a semigroup. If the condition B holds then $(X, \times)$ is a semigroup if and only if the conditions*

(i) u is not a θ-element; (ii) $x \neq u$ & ($x \neq u \times u$ or x is not a θ-element)

imply the conditions

(iii) if $u \star x \neq u$ and x is not a right unity of the semigroup $(X, \star)$ then $(u \star u) \star x = (u \times u) \star x$; (iv) if $x \star u \neq u$ and x is not a left unity of the semigroup $(X, \star)$ then $x \star (u \star u) = x \star (u \times u)$; (v) $(u \times u) \star u = u \star (u \times u)$.

The author does not know the necessary and sufficient conditions for B. However, some results may be obtained.

Lemma 2. *Let $(X, \star)$ and $(X, \times)$ be similar groupoids, and $(X, \star)$ be a semigroup. If the condition B holds, then we have, for each θ-element u:*

1) either such element x exists that $u = u \star x = u \times x$ (which is equivalent to $x = x \star u = x \times u$);

2) or such element x exists that $u = x \times x \neq x \star x$.

The following is possible for every θ-element u: $u = u \star x = x \star u$ and $x = u \star u = x \star x$ for some x. There exists a hypothesis that all of the cases for the condition B can be reduced to this variant.

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Elementary equivalence of automorphism groups of Abelian p-groups

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In the paper we consider elementary properties (i. e., properties expressible in first order terms) of the automorphisms group of Abelian p-groups.

The first who considered connection of elementary properties of the different models with elementary properties of derivative models was A.I. Maltsev in [1] in 1961. He proved that groups $G_n(K)$ and $G_m(L)$, where $G = GL, SL, PGL, PSL$ and $n, m \geq 3$, K, L — fields of characteristics 0, are elementary equivalent iff $m = n$ and fields K and L are elementary equivalent.

In 1992, this theory was continued with the help of ultraproduct construction and Keisler-Chang isomorphism theorem by K.I. Beidar and A.V. Mikhalev in [2], in which they found general approach to problems of elementary equivalence of different algebraic structures and generalized Maltsev theorem in the case of K and L are skewfields or associative rings.

Continuation of this research was made in 1998–2001 years in papers of E.I. Bunina, in which results of A.I Maltsev extended for unitary linear groups over skewfields and associative rings with involution, also for Chevalley groups over fields.

In 2000, V. Tolstikh in [3] considered connection of the second order properties for skew fields with the first order properties of automorphism groups of spaces of infinite dimension over the skewfields. In 2003, E. I. Bunina and A. V. Mikhalev considered connection of the second order properties of associative rings and the first order properties of categories of modules, endomorphism rings, automorphism groups and projective spaces of modules of infinite rank over the rings, see [4].

In [5] E. I. Bunina and A. V. Mikhalev discovered connection of second order properties of an Abelian p-group with first order properties of its ring of endomorphisms.

In this paper we discover connection of the second order properties of an Abelian p-group and the first order group properties of its group of automorphisms, for the case of $p > 2$:

Theorem 1. *Let A, C be p-groups, $p \geq 3$. If $\operatorname{Aut} A \equiv \operatorname{Aut} C$, then basic subgroups and divisible parts of the groups A and C are equivalent in second order logic*

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Characterizing Bezout rings of stable range n

O. Romaniv

Throughout this paper R will denote an associative ring with $1 \neq 0$.

A row $(a_1, a_2, \ldots, a_n)$ of elements of a ring R is a *right unimodular row* if there are elements $x_i \in R$, $1 \leq i \leq n$, with $\sum a_i x_i = 1$. If $x = (x_1, \ldots, x_n)$ is unimodular, then we say that x is *reducible* if there exists $y = (y_1, \ldots, y_{n-1})$ such that the $(n-1)$-row $(x_1 + x_n y_1, \ldots, x_{n-1} + x_n y_{n-1})$ is unimodular. R is said to have *stable range* $n \geq 1$ if n is the least positive integer such that every unimodular $(n+1)$-row is reducible. The concept of a ring of the stable range is left-right equivalent. [1]

By a right (left) Bezout ring we mean a ring in which all finitely generated right (left) ideals are principal, and by a Bezout ring a ring which is both right and left Bezout. [2]

Theorem 1. *A ring R is a right Bezout ring if and only if for any elements $a, b \in R$ there are elements $d \in R$ and $a_0, b_0, c_0 \in R$ with $a_0R + b_0R + c_0R = R$ such that*

$$(a, b, 0) = d(a_0, b_0, c_0).$$

Theorem 2. *A commutative Bezout ring R is a ring with finite stable range n if and only if for any elements $a_1, a_2, \ldots, a_n \in R$ there are an element $d \in R$ and a unimodular row $(a_1^0, a_2^0, \ldots, a_n^0)$ of elements of a ring R such that*

$$(a_1, a_2, \ldots, a_n) = d(a_1^0, a_2^0, \ldots, a_n^0).$$

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The purity of a module: a new invariant and its consequences

W. Rump

To any associative ring R with unity, we construct a site $\mathbf{mos}(R)$ which can be visualized as a tree (with infinite branches). Then R-modules are just abelian sheaves on this site. Since every sheaf on a tree is itself a tree, every module can be regarded as a tree. Based on this observation, we associate a cardinal invariant, the *purity*, to every R-module M, and show that the cardinality of M can be measured by means of its purity. As an application, we give a very general, quantitative existence theorem for flat covers and answer some related open questions.

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Groups of automata without cycles with exit

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Let X be a finite nonempty set. This set is called an alphabet and its elements are called letters. An automaton over alphabet X is a tuple $A = \langle X, Q, \varphi, \lambda \rangle$, where Q denotes the set of states, $\varphi : Q \times X \to Q$ is the transition function and $\lambda : Q \times X \to X$ is the output function.

Consider the set $X^* = \bigcup_{n \geq 1} X^n \cup \{\Lambda\}$ of all words over alphabet X. On this set one can define an operation of concatenation. The transition and output functions of an automaton $A = \langle X, Q, \varphi, \lambda \rangle$ can be extended to the set $Q \times X^*$ by the next formulas. For all $q \in Q$, $w \in X^*$ and $x \in X$

$$\varphi(q, wx) = \varphi(\varphi(q, w), x), \qquad \varphi(q, \Lambda) = q,$$
$$\lambda(q, wx) = \lambda(q, w)\lambda(\varphi(q, w), x), \qquad \lambda(q, \Lambda) = \Lambda.$$

Every state $q \in Q$ defines a map $f_q = \lambda(q, \cdot) : X^* \to X^*$. The automaton A is called invertible if all these maps are bijections.

The group of an invertible automaton $A = \langle X, Q, \varphi, \lambda \rangle$ is the group generated by the set $\{f_q : q \in Q\}$ ([1]).

A cycle in an automaton $A = \langle X, Q, \varphi, \lambda \rangle$ is a sequence of pairwise different states $q_1, q_2, \ldots, q_n \in Q$, $n \geq 1$, such that there exists a sequence of letters $x_1, x_2, \ldots, x_n \in X$ which satisfies equalities $\varphi(q_i, x_i) = q_{i+1}$, $1 \leq i \leq n$, and $\varphi(q_n, x_n) = q_1$. This cycle is called a cycle with exit if there exist i, $1 \leq i \leq n$, and $x \in X$ such that $\varphi(q_i, x) \notin \{q_1, q_2, \ldots, q_n\}$. In other case this cycle is called a cycle without exit. The group of a finite automaton without cycles with exit is finite ([2]).

Theorem 1. *Let $A = \langle X, Q, \varphi, \lambda \rangle$ be an automaton without cycle with exit over binary alphabet, $|Q| = n$ and G be the group generated by this automaton. Then $|G| \leq 2^{2^{n-1}}$.*

Theorem 2. *For arbitrary positive integer n there exists an n-state automaton over binary alphabet without cycles with exit such that the order of the group of this automaton is $2^{2^{n-1}}$.*

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Endomorphisms and conjugacy of Sushkevich in the semigroup $T(N)$

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Let $T(N)$ be the semigroup of all (fully defined) transformations over the set of positive integers N. In [1] A.K. Sushkevich has introduced subsemigroups $In(N)$ of injective transformations over N and $Sur(N)$ of surjective transformations over N. Evidently we have $In(N) \cap Sur(N) = S(N)$, where $S(N)$ is the symmetric group over N.

Lemma 1. *For any permutation $f \in S(N)$ there exist transformations $g \in In(N)$ and $h \in Sur(N)$ such that $f = gh$. The transformation g may be taken in arbitrary way and the transformation h for fixed g in a number of way.*

It follows from this lemma that for any $g \in In(N)$ there exists $h \in Sur(N)$ such that $Id = gh$ where Id is the identity permutation.

Definition 1. *A pair (g, h), $g \in In(N)$, $h \in Sur(N)$ is called admissible if $gh = Id$.*

Theorem 1. *For any admissible pair (g, h) the mapping $\varphi_{g,h} : T(N) \to T(N)$ which is defined by $\varphi_{g,h}(u) = hug$, $u \in T(N)$, is a monomorphism of the semigroup $T(N)$ in itself. Endomorphism $\varphi_{g,h}$ is an automorphism if and only if $g \in S(N)$, $h = g^{-1}$.*

Definition 2. *Transformations u, v are called conjugated in the sense of Sushkevich if there exist admissible pairs (g_1, h_1), (g_2, h_2) such that $u = g_1vh_1$, $h = g_2uh_2$.*

Theorem 2. *The conjugation in the sense of Sushkevich is an equivalence relation on the semigroup $T(N)$.*

We call equivalence classes for this relation conjugacy classes in Sushkevich sense in the semigroup $T(N)$. Problem of the characterization of this classes is a generalization of the problem of characterization of conjugacy classes in $S(N)$.

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Multilinear components of the prime subvarieties of the variety $Var(M_{1,1})$

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Consider the free associative algebra $F\langle X\rangle$ over a field F generated by a countable set X. A T-ideal Γ of the algebra $F\langle X\rangle$ is called a *verbally prime* iff for every T-ideals Γ_1, Γ_2 an inclusion $\Gamma_1\Gamma_2 \subseteq \Gamma$ implies one the inclusions $\Gamma_1 \subseteq \Gamma$ or $\Gamma_2 \subseteq \Gamma$. A variety of the algebras is called *prime* if its ideal of identities is verbally prime.

In the case of characteristic zero ($char F = 0$) the prime varieties were described by A.R. Kemer in [1]. The problem of classification of prime varieties in the case $char F = p > 0$ is open and very difficult.

Denote by G the Grassmann algebra with the standard $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$. Consider the algebra $M_2(G) = M_2 \otimes G$ and its subalgebra $M_{1,1}$:

$$M_{1,1} = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \middle| A, D \in G_0, B, C \in G_1 \right\}.$$

We describe the multilinear identities of a prime subvariety of the variety $Var(M_{1,1})$.

Theorem 1. *If $charF = p > 2$ then the set of multilinear identities of a prime subvariety of the variety $Var(M_{1,1})$ coincides with the set of multilinear identities of the algebra $M_{1,1}$, or it is generated by the identity $[x, y, z] = 0$, or it is generated by the identity $[x, y] = 0$.*

If $charF = 2$ then the multilinear identities of the varieties $Var(M_{1,1})$ and $Var(M_2)$ coincide. In this case the multilinear components of prime subvarieties of the variety $Var(M_{1,1})$ were described by A.R. Kemer in [2].

It is easy to see that the Theorem 1 follows from the Theorem 2:

Theorem 2. *Let $p \neq 2$, U be any T-ideal with the properties $T[M_{1,1}] \subset U$, $T[M_{1,1}] \cap P \neq U \cap P$. Then for some m*

$$[x_1, y_1, z_1][x_2, y_2, z_2] \dots [x_m, y_m, z_m] \in U.$$

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Zeta-function of k-form

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Let n, k be the positive integers, $k \geq 2$ and let $\alpha, \delta \in (0, 1]^n$. Consider the absolutely convergent series for $\Re s > 1$

$$\xi_{n,k}(s; \alpha, \delta) = \sum_{m \in \mathbb{Z}^n_{\geq 0}} e^{2\pi i m \cdot \alpha} \left(\sum_{j=1}^{n} (m_j + \delta_j)^k \right)^{-s},$$

(here $m \cdot \alpha$ denotes the dot product of m and α, $m = (m_1, \dots, m_n)$, $\delta = (\delta_1, \dots, \delta_n)$)
which we call the zeta-function of the k-form

$$Q(m, \delta) = \sum_{j=1}^{n} (m_j - \delta_j)^k$$

For $k = 2$, $n \geq 2$, we obtain the classic Epstein zeta-function, and for $k = n \geq 3$, $\alpha = \delta = (0, \dots, 0)$ the $\xi(s; \alpha, \delta)$ coincides with the Waring zeta-function introduced by A. Vinogradov[1].

We construct the integral representation of $\xi_{n,k}(s; \alpha, \delta)$ under condition $\frac{k}{n} < n \leq k$, which permits to study the behavior of this function in neighborhood of poles in the points $s = \frac{\ell}{k}$, $\ell = 0, 1, \dots, n$, and $s = 1$.
Moreover, we obtain the asymptotic formulae for the summatory function of the number of representations N as the sum of n k-th powers of non-negative integers.

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On the Lausch's Problem for π-Normal Fitting Classes

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All groups considered are finite and solvable. A Fitting class $\mathfrak{F}$ is said to be a maximal Fitting subclass of a Fitting class $\mathfrak{H}$ (denoted by $\mathfrak{F} < \cdot \mathfrak{H}$) if $\mathfrak{F} \subset \mathfrak{H}$ and the condition $\mathfrak{F} \subseteq \mathfrak{M} \subseteq \mathfrak{H}$, where $\mathfrak{M}$ is a Fitting class, always implies $\mathfrak{M} \in \{\mathfrak{F}, \mathfrak{H}\}$. A Fitting class $\emptyset \neq \mathfrak{F}$ is called $\mathfrak{X}$-normal (denoted by $\mathfrak{F} \trianglelefteq \mathfrak{X}$) if $\mathfrak{F} \subseteq \mathfrak{X}$ and for every $\mathfrak{X}$-group G its $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-maximal subgroup of G.

Definition. *Let $\mathbb{P}$ be the set of all primes and $\emptyset \neq \pi \subseteq \mathbb{P}$. If a Fitting class $\mathfrak{F}$ is normal in the Fitting class $\mathfrak{S}_\pi$ of all π-groups then we call $\mathfrak{F}$ π-normal.*

Theorem. *Every maximal Fitting subclass of a π-normal Fitting class is π-normal.*

Corollary 1. *(J. Cossey [1]). Let $\mathfrak{F}$ be a Fitting class. If $\mathfrak{F} < \cdot \mathfrak{S}$ then $\mathfrak{F} \trianglelefteq \mathfrak{S}$.*

Corollary 2. *[2]. Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes. If $\mathfrak{F} < \cdot \mathfrak{H} \trianglelefteq \mathfrak{S}$ then $\mathfrak{F} \trianglelefteq \mathfrak{S}$.*

Corollary 3. *[3]. Let $\mathfrak{F}$ be a Fitting class. If $\mathfrak{F} < \cdot \mathfrak{S}_\pi$ then $\mathfrak{F} \trianglelefteq \mathfrak{S}_\pi$.*

The intersection of all non-identity $\mathfrak{S}$-normal Fitting classes is a non-identity $\mathfrak{S}$-normal Fitting class $\mathfrak{S}_\star$ which is called the minimal normal Fitting class [4]. H. Lausch had formulated a problem (see question 9.18 [5]) on existence of maximal Fitting subclasses in the minimal normal Fitting class $\mathfrak{S}_\star$. It is proved (see corollary 2 [6]) that $(\mathfrak{S}_\pi)_\star$ is the unique non-trivial minimal π-normal Fitting class. Corollary 4 gives a negative answer to the Lausch's problem for π-normal Fitting classes.

Corollary 4. *The minimal π-normal Fitting class $(\mathfrak{S}_\pi)_\star$ has no maximal Fitting subclasses.*

Corollary 5. *(N. T. Vorob'ev [7]). The minimal normal Fitting class $\mathfrak{S}_\star$ has no maximal Fitting subclasses.*

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Frattini series and involutions of finite L-groups

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Two classes of finite 2-groups are explored: L - the class of all 2-groups G, every maximal cyclic subgroup of which has a supplement; H - the class of all finite 2-groups from L every homomorphic image of which is a group from L.

Fundamental results from a theory of L-group are laid out in the articles [1] [2]. In particular, in [1] it is set, that every finite 2-group G from L has exact factorization into cyclic subgroups. It is proved [2], that a class L is not closed under homomorphic images. Other results are about groups which can be factorized into transposite subgroups (see [3]).

In this research the groups from H, their Frattini series and central involution of these groups are explored. Let G be any L-group with exponent 2^{n+1} and Φ(G) its Frattini subgroup. Let $\Phi_0 =$ G, $\Phi_1 = \Phi$(G), $\Phi_{k+1} = \Phi(\Phi_k)$, k $\geq$ 1. In [1] important facts about subgroup Φ_k are proved.

Next theorems about subgroups from classes L and H are proved in this research.

Theorem 1. *Let G be any H-group and T its maximal cyclic subgroup from G, $N_G(T) = T \lambda S$ and $M = C_S(T)$. Then $C_G(T) = T \times M$ and $\overline{S} = S/M$ is cyclic.*

Theorem 2. *Let $G \in L$ and suppose there is normal supplement in G for all maximal cyclic subgroups $< g >$. Then G is abelian group.*

Theorem 3. *Let G be any L-group with exponent 2^k. Then there exists central involution u in group Φ_{k-1}, such that factor group $G/ <u> \in L$.*

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Automorphisms and elementary equivalence of the semigroup of invertible matrices with nonnegative elements over commutative rings

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Let R be an ordered ring, $G_n(R)$ the subsemigroup of $GL_n(R)$, consisting of all matrices with nonnegative elements. In [1] A.V. Mikhalev and M.A. Shatalova described all automorphisms of the semigroup $G_n(R)$ in the case when R is a linearly ordered skewfield and $n \geqslant 2$. In the work [2] E.I. Bunina and A.V. Mikhalev described all automorphisms of the semigroup $G_n(R)$, if R is an arbitrary linearly ordered associative ring with $1/2$, $n \geqslant 3$. In the paper [3] E.I. Bunina and A.V. Mikhalev found necessary and sufficient conditions for these groups to be elementary equivalent.

We consider semigroups of invertible matrices with nonnegative elements over commutative partially ordered rings with $1/2$. We prove two following theorems:

Theorem 1. *Every automorphism of the semigroups of invertible matrices with nonnegative elements $G_n(R)$, where R is a commutative partially ordered ring with $1/2$, $n \geq 3$, coincides with a composition of an inner automorphism (conjugation by some invertible in $G_n(R)$ matrix), a ring automorphism (induced by some automorphism of the semiring R_+ of nonnegative elements), a central homothety, on some special subsemigroup of $G_n(R)$ generated by elementary matrices.*

Theorem 2. *If the semigroups $G_n(R)$ and $G_m(S)$ (R, S are commutative partially ordered rings with $1/2$, $n \geqslant 3$) are elementary equivalent, then $m = n$ and the semirings R_+ and S_+ are elementary equivalent.*

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About groups close to hamiltonian

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Let G be a group. Let's denote with $\mathbf{L_{non-norm}}(G)$ the family of all non-normal subgroups of G. Study of the impact of $\mathbf{L_{non-norm}}(G)$ on the structure of group G was started long ago and continues to this day. G.M. Romalis and M.F. Syesekin in [1]-[3] started to study the groups, in which family $\mathbf{L_{non-norm}}(G)$ consists of abelian subgroups. Such groups were called metahamiltonian groups. The full description of metahamiltonian groups have been obtained in the monograph of N.F. Kuzennyj and N.N. Semko [4]. The natural continuation of such researches is a consideration of a situation, when the subgroups of the family $\mathbf{L_{non-norm}}(G)$ belong to the class of groups, which are a natural extension of the class of abelian groups (for instance, the subgroups of a family $\mathbf{L_{non-norm}}(G)$ have finite derived subgroups or are FC-groups). Since the Chernikov groups are a natural extension of the finite groups, then the groups with Chernikov derived subgroups are a natural extension of the groups with finite derived subgroups.

We start the study of groups, in which every subgroup either is normal or has a Chernikov derived subgroups.

Theorem 1. *Let G be a locally (soluble-by-finite) group whose subgroups either are normal or have Chernikov derived subgroups. Then the following assertions hold:*

(1) *Every finitely generated subgroup of G is central-by-finite, in particular, G is a locally FC-group.*

(2) *The derived subgroup of a group G is a locally finite subgroup. In particular, if G is torsion-free, then G is abelian.*

Theorem 2. *Let G be a locally graded group whose subgroups either are normal or have Chernikov derived subgroups. Suppose that G is not locally (soluble-by-finite) group. Then the following assertions hold:*

(1) *G contains a normal locally finite subgroup T with G/T to be a non-periodic abelian group.*

(2) *T does not have finite system of generated elements.*

(3) *Every proper subgroup of T has Chernikov derived subgroups.*

Theorem 3. *Let G be a locally soluble group. If every proper subgroup of G has Chernikov derived subgroup, then derived subgroup of whole group G is Chernikov subgroup.*

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Canonical matrices of bilinear and sesquilinear forms over finite extensions of the field of p-adic numbers

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Canonical matrices of
(a) bilinear and sesquilinear forms,
(b) pairs of forms in which every form is symmetric or skew-symmetric, and
(c) pairs of Hermitian forms
are given over fields that are finite extensions of the field of p-adic numbers $\mathbb{Q}_p$, $p \neq 2$.

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Supersymmetry classes of tensors

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In [1], a joint paper by me and A. Madadi, we introduced a concrete method to construct the irreducible representations of the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, using the notion of symmetry classes of tensors. A similar work can be done for the Lie superalgebra $\mathfrak{sl}(p|q)$, if we have a suitable notion of *supersymmetry classes of tensors*. On the other hand, recently the term *super linear algebra* has been widely appeared in both mathematics and physics articles which study objects of linear algebra from *super-structure* point of view, (see [3]). Since during last five decades, there have been published several articles concerning symmetry classes of tensor, it is natural to have in hand the *super-version* of this linear algebra object. This article is the first step toward introducing supersymmetry classes of tensor and the author hopes the resulting notion will be very interesting both from multilinear algebra and from representation of Lie superalgebras point of view.

In this paper, we introduce the notion of a *supersymmetry class of tensors* which is the ordinary symmetry class of tensors $V_\chi(G)$ with a natural $\mathbb{Z}_2$-gradation. We give the dimensions of *even* and *odd* parts of this gradation as well as their natural bases. Also we give a necessary and sufficient conditions for the odd or even part of a supersymmetry class to be zero.

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Generalized characteristic polynomial

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Let n be a positive integer number and G be a subgroup of the full symmetric group on n letters. Assume that F is a favorite field and c is a function from G to F. We refer the generalized matrix function afforded by G and c, d_c^G, which is a generalization of the concept of ordinary determinant of n by n matrices. By using d_c^G, we refer and determine the generalized characteristic polynomials of n by n matrices over a favorite filed F afforded by some permutation groups, which are the generalizations of the concept of ordinary characteristic polynomial.

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On classification of Chernikov p-groups

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Let M be a divisible abelian p-group with minimality condition and H be a finite p-group. In papers [1, 2, 3] the Chernikov p-groups $G(M, H, \Gamma)$, which are the extensions of the group M by the group H and which are defined by some matrix representation Γ of the group H over the ring $\mathbb{Z}_p$ of p-adic integers, has been studied using the theory of integral p-adic representation of finite groups. In particular all Chernikov p-groups of type $G(M, H, \Gamma)$ were classified up to isomorphism in the case if H is the cyclic p-group of order p^s and Γ runs the set of all $\mathbb{Z}_p$-representations of the group H which contains not more than k ($k \leq 3$) nonequivalent irreducible components $\Delta_1, \ldots, \Delta_k$, in which connection if $k = 3$ then Δ_1 is identity representation. It's also has been shown that the problem of the description up to isomorphism of all Chernikov groups of type $G(M, H, \Gamma)$, where Γ runs the set of all $\mathbb{Z}_p$-representations of the p-group H which contains k nonequivalent irreducible components is wild if one of the following conditions holds:

1) H is the cyclic p-group of order p^s, $s > 3$, $k > 4$;

2) H is the cyclic p-group of order p^s, $s > 2$, $k = 3$, $p > 3$ and the degree of the representation Δ_i greater then 1 ($i = 1, 2, 3$);
3) H is the cyclic p-group of order p^s, $s > 2$, $k > 3$, $p > 2$,
4) H is the noncyclic p-group, $k > 4$.

Recently we have obtained the classification up to isomorphism of some Chernikov p-groups of type $G(M, H, \Gamma)$ in the case if H is an abelian p-group and Γ runs the set of all indecomposable $\mathbb{Z}_p$-representations of the group H which contains precisely two nonequivalent irreducible components.

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On the invariants of polynomial matrices with respect to semiscalar equivalence

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In the present report, some class of polynomial matrices is studied in connection with their reducibility by semiscalar equivalent transformations to some simpler form. We consider related researches [1, 2] in which the conditions for semiscalar equivalence of polynomial matrices having only two distinct invariant factors are indicated. Let us now turn to matrices with three and more distinct invariant factors.

Assume that $N(x) \in M_n(C[x])$, $\det N(x) = x^k$ and the Smith form of the matrix $N(x)$ is of the form $diag(x^{k_1}, \ldots, x^{k_n})$, where $1 \le k_1 < \ldots < k_n$. By a *codegree* of the polynomial $a(x) \in C[x]$ we mean the lowest degree of its monomial and denote it by $co\deg\ a$. The coefficient of this monomial is called the *younger coefficient* of $a(x)$. By definition, $co\deg\ 0 = +\infty$. By a codegree of the polynomial row (column) we mean the lowest codegree of its elements. An element $a_i(x)$ of the row $\bar{a}(x) = \left\| \begin{matrix} a_1(x) & \ldots & a_n(x) \end{matrix} \right\|$ (column $\bar{a}^T(x)$) is said to by *key element* if $co\deg\ a_i = co\deg\ \bar{a}$ and $co\deg\ a_i < co\deg\ a_j$ for all j, $j > i$ ($j < i$). We shall say that two rows (columns) $\bar{a}_p(x)$ and $\bar{a}_q(x)$, $p < q$, *form an inversion* if $co\deg\ \bar{a}_p > co\deg\ \bar{a}_q$ ($co\deg\ \bar{a}_p < co\deg\ \bar{a}_q$), or $co\deg\ \bar{a}_p = co\deg\ \bar{a}_q$ and the key element of $\bar{a}_p(x)$ is arranged on the left (above) of the key element of $\bar{a}_q(x)$.

Theorem 1. *In the class $\{CN(x)Q(x)\}$, $C \in GL_n(C)$, $Q(x) \in GL_n(C[x])$ of semiscalarly equivalent matrices there exists a matrix of the form*

$$CN(x)Q(x) = \left\| \begin{matrix} x^{k_1} & 0 & \ldots & 0 \\ a_{21}(x) & x^{k_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(x) & a_{n2}(x) & \ldots & x^{k_n} \end{matrix} \right\| = F(x)\ ,\ (*)$$

that satisfies the following conditions:

1) $\deg a_{ij} < k_i$, $co\deg\ a_{ij} > k_j$, $i = 2, \ldots, n$, $j < i$*;*

2) each row has highest codegree;

3) the key elements of rows of the same codegree are arranged in the different columns;

4) the younger coefficients of the key elements of rows equal 1.

Theorem 2. *In the class of semiscalarly equivalent matrices of the form (*) with the 1) - 4) properties, there exists such a matrix, that its adjoint matrix* $F_*(x)$ *satisfies the following conditions:*

1) each column has highest codegree;

2) the key elements of two arbitrary columns of the same codegree are arranged in the different rows if the corresponding pair of rows of matrix $F(x)$ *does not form an inversion.*

Theorem 3. *The class* $\{CN(x)Q(x)\}$ *uniquely defines the positions and codegrees of the key elements of rows of matrix* $F(x)$ *and of columns of matrix* $F_*(x)$ *as well as younger coefficients of the key elements of columns of matrix* $F_*(x)$*.*

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On matrices reduction by one-side transformations

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In studying some matrix problems, in particular, the factorization of matrices one needs to describe all non-associate matrices with prescribed canonical diagonal form. The classical Hermite normal form is improper to this problem because it describes non-associate matrices with prescribed determinant. Our purpose is to provide the first step toward the construction of normal form of such matrices by one-side transformations, which clarify the canonical diagonal form of the matrix.

Let R be an adequate domain and A be $n \times n$ matrix over R with canonical diagonal form $E_t \oplus \varphi E_{n-t}$, $\varphi \neq 0$, $1 \leq t < n$. Then there exist matrices P, Q such that

$$PAQ = E_t \oplus \varphi E_{n-t} = \Phi \ .$$

Theorem. *Let* $P = \left\| \begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right\|$, *where* P_{11} *is* $t \times t$ *matrix and*

$$\left\| \begin{array}{cccc} \beta_1 & 0 & & 0 \\ * & \beta_2 & & 0 \\ & & \ddots & \\ * & * & & \beta_n \end{array} \right\|$$

be left Hermite normal form of the matrix $(\varphi E_t \oplus E_{n-t})P$. Then there exists matrix U such that

$$AU = \left\| \begin{matrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{matrix} \right\|^{-1} \Phi,$$

where

$$C_{22} = \left\| \begin{matrix} \beta_{t+1} & 0 & & 0 \\ c_{t+2.t+1} & \beta_{t+2} & & 0 \\ \vdots & & \ddots & \\ c_{n.t+1} & c_{n.t+2} & & \beta_n \end{matrix} \right\|,$$

where c_{ij} lie in complete set of residues modulo β_j, $i = t+2, t+3, \ldots, n$, $j = t+1, t+2, \ldots, n-1$. The entries c_{ij}, β_k, $k = t+1, t+2, \ldots, n$, are uniquely determined independently of the choice of P.

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Asymptotic dimension of linear type and functors in the asymptotic category

O. Shukel'

Let (X, d) be a metric space. Recall that the asymptotic dimension of X does not exceed n (written $\operatorname{asdim} X \le n$) if for every $D > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X such that $\mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i$, where every $\mathcal{U}_i$ is a D-discrete family (see [1]).

The notion of asymptotic dimension plays an important role in the geometric group theory and another areas of mathematics.

The following notion, which is a modification of the asymptotic dimension, is introduced in [2]. The asymptotic dimension of linear type of a metric space X does not exceed n, if there is $c > 0$ such that, for every $D > 0$, there exist $D' > D$ and a cover $\mathcal{U}$ of X which satisfy the properties:

1) $\mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i$, where every family $\mathcal{U}_i$ is D'-discrete;

2) $\operatorname{mesh}(\mathcal{U}) < cD'$, where $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam}(U) | U \in \mathcal{U}\}$.

The author [3] proved that some covariant functors in the asymptotic category (see [4]) preserve the class of metric spaces of finite asymptotic dimension. The aim of this talk is to extend these results over the case of spaces of finite asymptotic dimension of linear type. Moreover, we obtain some exact estimates similar to those in [5].

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On Krull dimension of serial piecewise Noetherian rings

Vjacheslav V. Shvyrov

One property of right Noetherian rings and Noetherian modules is that they have a Krull dimension in the sense Rentschler-Gabriel. Naturally arises the question of having Krull dimension for the case of serial piecewise Noetherian rings. A positive answer to this question is obtained.

Definition 1. *A ring A is called a right piecewise Noetherian ring if it satisfies the following conditions:*

(i) A is a finite set of pairwise orthogonal primitive idempotents $e_1, e_2, \dots, e_n$ such that $1 = e_1 + e_2 + \dots + e_n$;

(ii) eAe is right Noetherian ring for any primitive idempotent $e \in A$.

Definition 2. *If a, b belong to a poset A, and $a \geq b$, then we define $a/b = \{x \in A \mid a \geq x \geq b\}$. This is a subposet of A and is called the factor of a by b. By descending chain $\{a_n\}$ of elements of A we mean that $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots$; and the factors a_i/a_{i+1} are called the factors of the chain.*

Definition 3. *We define the deviation of a poset A, $devA$ for short. If A is trivial then $devA = -\infty$. If A is nontrivial but satisfies the d.c.c. then $devA = 0$. For a general ordinal α, we define $devA = \alpha$ provided:*

(i) $devA \neq \beta < \alpha$;

(ii) in any descending chain of elements of A all but finitely many factors have deviation less than α.

Definition 4. *If M is right module over ring R then Krull dimension of M, written $K\dim(M)$, is defined to be the deviation of $L(M)$, the lattice of submodules of M. In particular, $K\dim(R_R)$ is the right Krull dimension of R.*

Lemma 1. *If M_R is Noetherian then $K\dim(M)$ exists; and if R is right Noetherian then $K\dim(R_R)$ exists.*

Theorem 1. *Let R be a serial piecewise Noetherian ring and $1 = e_1 + e_2 + \dots + e_n$, then R has Krull dimension and $K\dim(R) \geq m$ iff $K\dim(R_i) \geq m$, m is natural, $i = 1, \dots, n$, where $R_i = e_iRe_i$.*

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Local Functions of Fitting Classes

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All groups considered are finite. In definitions and notations we follow [1].

Let $\mathbb{P}$ be the set of all prime numbers. A map $f : \mathbb{P} \to \{\text{classes of groups}\}$ is called a local function [2]. Note that if all values of a local function f are Fitting classes, then f is called an H-function or Hartley function. Let $\pi = \mathrm{Supp}(f)$ and $\mathfrak{E}_\pi$ be the class of all π-groups. If there exists an H-function f such that $\mathfrak{F} = \mathfrak{E}_\pi \cap (\cap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'})$ then the class $\mathfrak{F}$ is called local [3].

It was proved [4] that every local Fitting class is defined by the largest integrated H-function F such that all nonempty values of F are Lockett classes [1] and $F(p) = F(p)\mathfrak{N}_p \subseteq \mathfrak{F}$ for all prime p.

In this paper a new local definition for an arbitrary Fitting class $\mathfrak{F}$ by a description of its local function is shown.

Definition. *Let $\mathfrak{X}$ and $\emptyset \neq \mathfrak{F}$ be Fitting classes. We define the class of groups $\mathfrak{F}_\mathfrak{X}$ as follows: $G \in \mathfrak{F}_\mathfrak{X}$ if and only if the $\mathfrak{F}$-radical of G is an $\mathfrak{X}$-group. If $\mathfrak{X} = \emptyset$ then $\mathfrak{F}_\mathfrak{X} = \emptyset$.*

Theorem. *Every local Fitting class $\mathfrak{F}$ is defined by the local function x such that:*

$$x(p)\mathfrak{N}_p = x(p) = \begin{cases} \mathfrak{F}_{F(p)}, & \text{if } p \in \pi; \\ \emptyset, & \text{if } p \in \pi', \end{cases}$$

where F is the largest integrated H-function of the class $\mathfrak{F}$ and $\pi = Supp(F)$.

The theorem implies some corollaries which give us descriptions of new local definitions of known Fitting classes. In particular, we have the following

Corollary. *The Fitting class $\mathfrak{N}$ of all nilpotent groups is defined by the local function x such that for every prime p*

$$x(p)\mathfrak{N}_p = x(p) = (G : G_\mathfrak{N} \in \mathfrak{N}_p).$$

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On zeta-functions associated to certain cusp forms

D. Šiaučiūnas

Let $F(z)$ be a normalized Hecke eigenform of weight κ for the full modular group with the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}, \quad c(1) = 1.$$

The zeta-function $\varphi(s, F)$, $s = \sigma + it$, attached to F is defined, for $\sigma > \frac{\kappa+1}{2}$, by

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and is analytically continuable to an entire function.

Let $w \neq 0$ be an arbitrary complex number. Then, for $\sigma > \frac{\kappa+1}{2}$,

$$\varphi^w(s, F) = \sum_{m=1}^{\infty} \frac{g_w(m)}{m^s},$$

where $g_w(m)$ is a multiplicative function. Let $h(m) = g_w^2(m) m^{1-\kappa}$. Then in [1], an asymptotic formula for the mean value of

$$\sum_{m \leq x} h(m), \quad x \to \infty, \tag{1}$$

in the case $|w| \leq \frac{1}{2}$ and $\mathrm{Re} w^2 > 0$ has been obtained. In the report, we will discuss the case $|w| > \frac{1}{2}$.

Asymptotic formula for the mean value of (1) is applied in the investigation of the moments

$$\int_0^T |\varphi(\sigma + it, F)|^{2k} \mathrm{d}t$$

for $\sigma \geq \frac{\kappa}{2}$, $k \geq 0$ and $T \to \infty$.

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On HM^*-groups

L. V. Skaskiv

It is joint work with O.D. Artemovych. Let p be a prime. Recall that a group G is called an HM^*-group if its commutator subgroup G' is hypercentral and the quotient group G/G' is a divisible Černikov p-group (see [1] and [2]). Any group of Heineken-Mohamed type is an HM^*-group. It is obvious that a hypercentral HM^*-group is a divisible Černikov p-group. Any locally nilpotent HM^*-group is a p-group. If the commutator subgroup G' of HM^*-group G has no proper supplements in G, then G is also a p-group. Recall that a group G is called indecomposable if any two proper subgroups generate a proper subgroup in G.

Theorem. *Let G be an HM^*-group.*

(1) If G is an indecomposable group, then every proper subgroup of G is hypercentral.

(2) Suppose that G is a soluble group with the normalizer condition. Then

(i) G/G' is a quasicyclic p-group if and only if G is indecomposable;

(ii) if the commutator subgroup G' is nilpotent of finite exponent, then all subgroups are subnormal in G.

We study also other properties of HM^*-groups and construct some examples of HM^*-groups.

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On one property of the lattice of all n-multiply ω-composition formations

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All groups considered are finite. The notations and terminology see in [1], [2].

Recall that a complete lattice of formations Θ is called inductive [1] if for any set of Θ^l–formations $\{\mathfrak{F}_i \mid i \in I\}$ and for any set of inner Θ-valued satellites $\{f_i \mid i \in I\}$ [2], where f_i is a satellite of $\mathfrak{F}_i$, we have

$$\vee_{\Theta^l}(\mathfrak{F}_i \mid i \in I) = LF(\vee_{\Theta}(f_i \mid i \in I)).$$

Theorem. *The lattice of all n-multiply ω-composition formations is inductive.*

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A criterion for two-modality of ideals for one-branch one-dimensional singularities of type W

R. Skuratovski

One-branch one-dimensional singularity is, by definition, a subalgebra A in the ring of formal power series $S = K[[t]]$ over a filed K, such that S is a finitely generated A-module. They say that such a singularity is *of type* W, if A contains an element of order 4, i.e. an element $at^4 + o(t^4)$, where $a \neq 0$ is an element from the field K. A singularity is called *plane*, if the maximal ideal of the ring A is generated by 2 elements. Any one-branch singularity of type W always contains an element of order m not divisible by 4. We denote by $m(A)$ the smallest of such orders. We deduce a necessary and sufficient conditions in order that an one-branch singularity of type W has at most two-parameter families of ideals.

Theorem 1. *An one-branch singularity A of type W has at most 2-parameter families of ideals if and only if $m(A) \leq 11$.*

Equivalent condition: *A contains a plane singularity of type W_{24}, W_{30} or $W^{\#}_{2,2q-1}$ in the Arnold's classification* [1].

For such rings a complete description of ideals is given.

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About classification and solving Moufang-like functional equations on quasigroups

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A binary operation f, defined on a set Q, is said to be *invertible* or *quasigroup*, if every of the equations $f(x;a) = b$, $f(a;y) = b$ has a unique solution for all a, b of Q. Assigning x and y to every pair $(a;b)$ defines invertible operations f^ℓ and f^r on Q. So, the superidentities

$$F(F^\ell(x;y);y) = x, \quad F(x;F^r(x;y)) = y, \quad (F^\ell)^\ell = F, \quad (F^r)^r = F \tag{1}$$

hold, i.e. the equalities are true for all $F \in \Delta$ and for all $x, y \in Q$, where Δ denotes the set of all quasigroup operations of Q.

Functional equations, having no functional and subject constants and having two-placed functional variable only, are under consideration. An equation is called *general*, if all functional variables are pairwise different. Two functional equations are said to be *parastrophic equivalent* (see [1]), if one can be obtained from the other in a finite number of renaming functional or subject variables or applying the superidentities (1).

Solving of the Moufang functional equations is a well-known problem in the theory of quasigroups [2]. Every of these three equations as well as Bol functional equation has two subject two-appearence variables and one four-appearence variable.

Theorem 1. *Every general functional equation, having two subject two-appearence variables and one four-appearence variable is parastrophic equivalent to at least one of eight functional equations.*

We have solved seven of the eight functional equations. For example,

Theorem 2. *A sequence $(f_1, \dots, f_6)$ of quasigroup operations defined on Q is a solution of*

$$F_1(x; F_2(x; F_3(x; y))) = F_4(F_5(x; z); F_6(z; y)) \tag{2}$$

if and only if there exists a group $(Q; +)$ and substitutions α, β, γ, δ, ν, φ such that

$$\begin{gathered} f_3(x; y) = f_2^r(x; f_1^r(x; \varphi^{-1}(\delta x + \gamma y))), \quad f_4(x; y) = \varphi^{-1}(\alpha x + \nu y), \\ f_5(x; z) = \alpha^{-1}(\delta x - \beta z), \quad f_6(z; y) = \nu^{-1}(\beta z + \gamma y). \end{gathered} \tag{3}$$

Corollary. *A sequence $(f_1, \dots, f_6)$ of quasigroup operations, which are topological on the topological line $\mathbf{R}$ with the ordinary topology, is a solution of* (2) *if and only if there exist homeomorphisms α, β, γ, δ, ν, φ of the space $\mathbf{R}$ such that* (3) *hold, where* $(+)$ *is the additive operation of the field $\mathbf{R}$.*

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3-torsion of the Brauer group of elliptic curve with additive reduction over general local field

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In a series of papers (see. e.g., [1] and references there) V.I. Yanchevskiĭ, V.I. Chernousov, V.I. Guletskii, G.L. Margolin, U. Rehman, S.V. Tikhonov described the 2-component of the Brauer group of an elliptic or hyperelliptic curve defined over a local field of characteristic zero. The study of 3-primary torsion of the Brauer group of an elliptic curve over a local field was initiated by V.I. Yanchevskiĭ and S.V. Tikhonov by investigating the case of elliptic curves with additive reduction [2]. Earlier it was shown by the author that some results concerning 2-torsion of the Brauer group can be extended to the case of elliptic curves over complete discretely valued fields with pseudofinite residue fields.

The purpose of this talk is to show that the 3-component of the Brauer group of an elliptic curve with additive reduction defined over a general local field [3] may be described in the same way as in the case of local ground field. Namely, it is proved that 3-primary torsion component of the Brauer group of an elliptic curve with additive reduction defined over general local field k is isomorphic either to the direct sum of 3-primary torsion component of the Brauer group of k and cyclic group of order 3 or to 3-primary torsion component of the Brauer group of k. Also, in the first case we obtain the representation of elements of the second summand of above mentioned direct sum by cyclic algebras. More precisely

Theorem. *Let E be the elliptic curve with additive reduction over general local field k, $\mathrm{char}\, k \neq 3$. If ${}_{3^n}\mathbf{Br}E \cong {}_{3^n}\mathbf{Br}(k) \oplus \boldsymbol{Z}/3$, E is given by Weierstrass equation:*

$$Y^2 = X^3 + (AX + B)^2,$$

then ${}_{3^n}\mathbf{Br}E$ is generated by the classes of the following unramified division algebras: $D_{3^n} \otimes_k k(E)$, where D_{3^n} is central over k division algebra of index 3^n, and $H = (Z_3(E), \sigma, h)$, where $h = \sqrt{f(X)} + AX + B$, $f(X) = X^3 + (AX+B)^2$, $Z_3(E)$ is unramified extension of $k(E)$ of degree 3 and σ is appropriate Frobenius automorphism.

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Порівняння у розумінні категорії сукупності вільних піднапівгруп напівгрупи цілих лінійних перетворень

М. І. Сумарюк

У роботах [1-4] встановлено, що у розумінні категорії або міри "більшість" підгруп (піднапівгруп) тієї чи іншої групи (напівгрупи) є вільними.

Ми розглядаємо напівгрупу $H(\mathbb{P})$ усіх цілих лінійних перетворень деякого числового поля $\mathbb{P}$, і нехай $H^k(\mathbb{P})$ – її довільний k-й ($k \in \mathbb{N}$) декартів степінь. Нехай $F_k(\mathbb{P})$ – множина усіх кортежів степеня $H^k(\mathbb{P})$, які породжують вільну піднапівгрупу напівгрупи $H(\mathbb{P})$. Для довільної фіксованої обмеженої підмножини X, $|X| > 1$, поля Р на степені $H^k(\mathbb{P})$ визначимо віддаль $d_{k,X}$, де для довільних двох впорядкованих наборів $\varphi = (f_1, f_2, ..., f_k)$ та $\psi = (g_1, g_2, ..., g_k)$ із $H^k(\mathbb{P})$, маємо:

$$d_{k,X}(\varphi, \psi) = \max_{1 \le j \le k} \sup_{z \in X} |f_j(z) - g_j(z)|.$$

У результаті отримаємо метричний простір $(H^k(\mathbb{P}), d_{k,X})$.

Нехай $N_k(\mathbb{P}) = H^k(\mathbb{P}) \backslash F_k(\mathbb{P})$. У випадку, коли $\mathbb{P}$ – зліченне числове поле, маємо таке твердження.

Теорема 1. *Множини $F_k(\mathbb{P})$ та $N_k(\mathbb{P})$ є множинами першої категорії Бера.*

Якщо $\mathbb{P}$ – незліченне числове поле таке, що дійсне поле $\mathbb{R}$ є його підполем, то у цьому випадку правильними є наступні твердження.

Лема 1. *Множина $F_k(\mathbb{P})$ є всюди щільною в $H^k(\mathbb{P})$.*

Лема 2. *Множина $F_k(\mathbb{P})$ є множиною другої категорії Бера.*

Лема 3. *Множина $N_k(\mathrm{P})$ – ніде не щільна в $H^k(\mathbb{P})$.*

На основі лем 1-3, у випадку поля $\mathbb{P}$ такого, що $\mathbb{R} \subseteq \mathbb{P}$, отримується таке твердження.

Теорема 2. *Більшість k-породжених піднапівгруп напівгрупи $H(\mathbb{P})$ є вільними у категорному розумінні Бера.*

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On the structure of quantum channels

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Let H and K be finite dimensional Hilbert spaces. In the literature of quantum information theory, a quantum channel from $\mathfrak{B}(H)$ to $\mathfrak{B}(K)$ is described as a linear map

$$\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)' \tag{1}$$

from the dual of $\mathfrak{B}(H)$ to the dual of $\mathfrak{B}(K)$ which holds property: linear map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(K)$ defined by an equation $\psi = \phi'$ (ψ is the adjoint map to the map ϕ) is a unital completely positive map.

Our aim is to describe the structure of a quantum channel from $\mathfrak{B}(H)$ to $\mathfrak{B}(K)$.

Suppose $\rho \in \mathfrak{B}(H)'$ and a is an arbitrary element of $\mathfrak{B}(H)$; then define $\delta\rho \in \mathfrak{B}(H)$ by the equation

$$\rho(a) = \mathrm{Tr}(a \cdot \delta\rho) \tag{2}$$

Vice versa suppose $s \in \mathfrak{B}(H)$ and a is an arbitrary element of $\mathfrak{B}(H)$; then define $\sum[s] \in \mathfrak{B}(H)'$ by equation

$$\sum[s](a) = \mathrm{Tr}(a \cdot s) \tag{3}$$

Proposition 1. *Let $\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)'$ be a linear operator. Then*

$$\psi(\rho) = \sum_{m,n=1}^{\dim(H)} \mathrm{Tr}((\delta\rho)e_{mn}) \sum[z_{nm}(\psi)]$$

where

1) $\{e_{mn} \mid m,n = 1,\ldots,\dim(H)\}$ is a family of matrix units in $\mathfrak{B}(H)$,

2) $z_{nm}(\psi) = \delta(\psi(\sum[e_{mn}]))$ for $m,n = 1,\ldots,\dim(H)$.

Theorem 1. *Let $\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)'$ be a linear operator; then ψ is a quantum channel iff next two conditions are held*

1) suppose $\{a_n \mid n = 1,\ldots,\dim(H)\}$ be a family of elements from $\mathfrak{B}(K)$; then

$$\sum_{m,n=1}^{\dim(H)} \mathrm{Tr}(a_m z_{nm}(\psi) a_n^*) \geq 0;$$

2) for all $m,n = 1,\ldots,\dim(H)$ equality $\mathrm{Tr}(z_{mn}(\psi)) = \delta_{mn}$ is held.

Corollary 1. *Let $\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)'$ be a linear operator, $Z \in \mathfrak{B}(H) \otimes \mathfrak{B}(K)$ and $Z = \sum_{m,n=1}^{\dim(H)} e_{mn} \otimes z_{nm}(\psi)$. ψ is a quantum channel if and only if $Z \geq 0$ and $\mathrm{Tr}_K(Z) = 1$, where $\mathrm{Tr}_K(a \otimes b) = \mathrm{Tr}(b)a$ for all $a \in \mathfrak{B}(H)$ and $b \in \mathfrak{B}(K)$.*

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Nullstellensatz over quasifields

D. V. Trushin

We investigate the least studied class of differential rings – the class of differential rings of nonzero characteristic. Namely, we develop geometrical theory of differential equations in nonzero characteristic.

Simple differential rings were studied by Yuan [1] and latter by Keigher [2] and [3], the last one introduces the notion quasifield. We define the notion of differentially closed quasifield and classify all such quasifields up to isomorphism. Main examples of quasifields are the Hurwitz series rings. We prove that differentially closed quasifields are the Hurwitz series rings over more than countable algebraically closed fields.

We prove a variant of Nullstellensatz for an arbitrary differentially closed quasifield. It allows us to define the notion of quasivariety as a set of all common zeros for some system of differential equations over differentially closed quasifield. The notion of regular mapping is defined in the same manner as in algebraic geometry.

Topological properties of quasivarieties coincide with that of maximal spectra of countably generated algebras over fields. Therefore quasivarieties are not necessarily noetherian, but there is a weaker analogue for them. All quasivarieties are ω-noetherian, in other words, every more than countable descending chain of closed subsets is stable.

A topological space is said to satisfy the Baire property if the intersection of any countable family of dense open sets is not empty. A subset is called to be big enough if it contains the mentioned intersection. The important fact that the Baire property holds for irreducible quasivarieties. In terms of the Baire property can be described the image of a regular mapping. Namely, if the closure of the image is irreducible, then the image is big enough in its closure.

Presented machinery can be regarded as a basis for reduction modulo p for differential equations.

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Recursive paths of quivers

S. Tsiupii

We introduce the concept of a recursive path of a quiver and use it for the study of exponent matrices. A notion of an exponent matrix is arisen from Ring theory. Some rings are given by exponent matrices, for example, tiled orders are. Exponent matrices are applied also in other mathematical theories, for example, in encoding theory. Each exponent matrix can be associated with some graph called a quiver and we can study such matrices by methods of Graph theory.

We begin with some notation.

Let $M_n(Z)$ be a ring of square $n \times n$-matrices over the ring of integers.

A matrix $\mathcal{E} = (\alpha_{ij})$ from the ring $M_n(Z)$ is called *an exponent matrix* if the following conditions hold:

(i) $\alpha_{ii} = 0$ for all $i = 1, 2, \ldots, n$;

(ii) $\alpha_{ik} + \alpha_{kj} \geq \alpha_{ij}$ for all $i, j, k = 1, 2, \ldots, n$.

An exponent matrix $\mathcal{E} = (\alpha_{ij})$ is called *reduced* if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i \neq j$.

Let $\mathcal{E}$ be a reduced exponent matrix, E be the identity one. Denote $\mathcal{E}^{(1)} = \mathcal{E} + E = (\beta_{ij})$, $\mathcal{E}^{(2)} = (\gamma_{ij})$, where

$$\gamma_{ij} = \min_k \{\beta_{ik} + \beta_{kj} - \beta_{ij}\}.$$

A graph Q is *a quiver* of an exponent matrix $\mathcal{E}$ if the adjacency matrix of Q is equal to $\mathcal{E}^{(2)} - \mathcal{E}^{(1)}$.

Now we introduce the concept of a recursive path.

Definition 1. *A simple path L_n at a quiver Q will be called recursive if the following conditions hold:*

(i) all vertices of the quiver Q belong to L_n;

(ii) the quiver Q has no arrows from any vertex of L_n to all following vertices of L_n except for next one.

Theorem 1. *(Criterion on a triangularity of an exponent matrix). An exponent matrix $\mathcal{E}$ is triangular up to an equivalence if and only if its quiver $Q(\mathcal{E})$ has a recursive path.*

We describe all recursive paths at a quiver.

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Index of exponent matrix

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A notion of an exponent matrix is arisen from Ring theory. Some rings are given by exponent matrices, for example, tiled orders are [1]. Exponent matrices are used also in applied task, for example, by planning multi factor experiments, by reliability test of different systems et cetera.

Let $M_n(Z)$ be a ring of square $n \times n$-matrices over the ring of integers.

Definition 1. *A matrix* $\mathcal{E} = (\alpha_{ij}) \in M_n(Z)$ *is called* an exponent matrix *if the following conditions hold:*

(i) $\alpha_{ii} = 0$ *for all* $i = 1, 2, \ldots, n$;

(ii) $\alpha_{ik} + \alpha_{kj} \geq \alpha_{ij}$ *for all* $i, j, k = 1, 2, \ldots, n$.

An exponent matrix $\mathcal{E} = (\alpha_{ij})$ is called *reduced* if $\alpha_{ij} + \alpha_{ji} > 0 \quad \text{for all} \quad i \neq j$.

Each exponent matrix can be related with some graph called a quiver.
Let $\mathcal{E}$ be a reduced exponent matrix, E be the identity one. Denote

$$\mathcal{E}^{(1)} = \mathcal{E} + E = (\beta_{ij}),$$

$$\mathcal{E}^{(2)} = (\gamma_{ij}), \quad \text{where} \quad \gamma_{ij} = \min_k \{\beta_{ik} + \beta_{kj} - \beta_{ij}\}.$$

A graph $Q(\mathcal{E})$ is called *a quiver* of an exponent matrix $\mathcal{E}$ if

$$[Q(\mathcal{E})] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)},$$

where $[Q(\mathcal{E})]$ is the adjacency matrix of $Q(\mathcal{E})$.

Definition 2. *The maximal real eigenvalue of adjacent matrix of the quiver* $Q(\mathcal{E})$ *is called* an index *of the exponent matrix* $\mathcal{E}$ *and denoted by* $in\,\mathcal{E}$.

Theorem 1. *For any integers* m *and* n $(0 < m \leq n)$ *there exists a reduced exponent matrix* $\mathcal{E}_n$ *such that*

$$in\,\mathcal{E}_n = m,$$

where n *is the order of the matrix* $\mathcal{E}_n$.

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Global dimension of semiperfect and semidistributive rings

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Let A be a prime right Noetherian semiperfect and semidistributive ring with nonzero Jacobson radical R. Following [1, Ch. 14] a ring A is called a tiled order.

Denote by $Q(A)$ the quiver of a tiled order A.

Theorem 1. *If* $gl.dim\,A$ *is finite, then* $Q(A)$ *is the strongly connected quiver without loops.*

Let $\mathcal{O} = k[[x]]$ be a ring of formal power series over the field k with the prime element $x = \pi$. Consider $A = \sum_{i,j=1}^{4} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the e_{ij} are the matrix units of $M_4(\mathcal{O})$. Let $(\alpha_{ij}) \in M_4(\mathbb{Z})$ has the following form: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 4 & 4 & 1 & 0 \end{pmatrix}$. The quiver $Q(A)$ is the following:

i.e., $Q(A)$ has no loops and $gl.dim\,A$ is infinite.

Let s be the number of vertices of $Q(A)$. We give a list of tiled orders A with finite global dimension for $1 \leq s \leq 5$.

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The semigroups of correspondence of groups

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Let A be a universal algebra. If subalgebra of $A \times A$ is considered as a binary relation on A, then the set $S(A)$ of all subalgebras from $A \times A$ is a semigroup under the products of relations [1]. Semigroup $S(A)$ is called a semigroup of correspondence of algebra A. Each endomorphism $\varphi : A \longrightarrow A$ is naturally interpreted as subalgebra of $A \times A$, which gives a natural immersion $EndA \hookrightarrow S(A)$. We shall also notice that $S(A)$ contains all congruences of algebra A. Some properties of semigroup $S(A)$ are described in [2].

We study semigroup $S(A)$ when A is a finite group.

Theorem 1. *If C_n is a cyclic group of order n, then*

$$|S(C_n)| = \sum_{k,l|n} g.c.d.(k,l).$$

Theorem 2. *If D_n is a dihedral group of degree n, then*

$$|S(D_n)| = \sum_{k,l|n} (g.c.d.(k,l) + \frac{n^2}{kl}\left(1 + \sum_{d|g.c.d.(k,l)} d \cdot \varphi(d)\right) + 2\tau(n)\sigma(n),$$

when n is odd.

$$|S(D_n)| = \sum_{k,l|n} (g.c.d.(k,l) + 2n \cdot \left[\sum_{k|n} \frac{\tau(n)}{k} + \sum_{k|n} \frac{\tau(\frac{n}{2})}{k} + \sum_{k|\frac{n}{2}} \frac{\tau(\frac{n}{2})}{k}\right] +$$

$$+\frac{n^2}{kl}\left(1 + \sum_{d|g.c.d.(k,l)} d \cdot \varphi(d)\right) + 12 \sum_{\substack{k,l|n,\\ k \equiv l \equiv 0(2)}} \frac{n^2}{kl} + 4 \sum_{\substack{k,l|n \\ k \equiv 0(2) \\ l \equiv 0(2)}} \frac{n^2}{kl},$$

when n is even.

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On Quivers of Serial Rings

H. V. Usenko

All rings are associative with $1 \neq 0$. Let A be a serial ring with the Jacobson radical R. Denote by $Q(A)$ the Gabriel quiver of A and by $PQ(A)$ the prime quiver of A [1, Ch. 11].

A ring A is decomposable if A is a direct product of two rings, otherwise A is indecomposable.

Theorem 1. *Let A be a right Noetherian serial indecomposable ring and $Q(A) = PQ(A)$. Then A is right and left Artinian. Conversely, let A be a semiprimary serial ring, then A is two-sided Artinian and $PQ(A) = Q(A)$.*

Let $\Gamma(A)$ be a Pierce quiver of a serial ring A [1, Ch. 11].

Theorem 2. *Let A be a serial indecomposable ring and $Q(A) = \Gamma(A)$. Then A is two-sided Artinian and $R^2 = 0$.*

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Exponential sums on the sequences of inversive congruential pseudorandom numbers

S. Varbanets

Let $p > 2$ be a prime number, $n \geq 2$ be a positive number.
H. Niederreiter and I. Shparlinski[1] investigated the inversive congruential pseudorandom numbers $\{x_k\}$ that generated by the recursive relation

$$x_{k+1} \equiv a x_k^{-1} + b (mod\ p^n), \tag{1}$$

where $(a,p) = 1$, $b \equiv 0 (mod\ p)$, $(x_0, p) = 1$, $x_k x_k^{-1} \equiv 1 (mod\ p^n)$.

In our work we study some exponential sums on the sequence of inversive congruential pseudorandom numbers $\{y_k\}$ defined by the recursion

$$y_{k+1} \equiv a y_k^{-1} + b + c y_k (mod\ p^n),\ (y_0, p) = 1,\ k = 0, 1, 2, \ldots, \tag{2}$$

where $p > 2$ is a prime number, $a, b, c, x_0 \in \mathbb{Z}$, $(a,p) = 1$, $b \equiv c \equiv 0 (mod\ p)$, $bc \equiv 0 (mod\ p^n)$.

The sequence $\{y_k\}$ is a particular case of the sequence $\{x_k\}$ defined by (1).

We proved that y_k can be represented as the polynomials of k with the coefficients depending on y_0: *if $bc \equiv 0\,(mod\ p^n)$, $\nu_p(b^{\ell-1}) < n \leq \nu_p(b^{\ell})$, $\ell \geq 3$, then*

$$\begin{aligned} y_{2k-1} &= kb + kcy_0 + (a + b^2 f_1(k)) y_0^{-1} + b f_2(k) y_0^{-2} + \\ &+ ((k-1)ca^2 + b^2 f_3(k)) y_0^{-3} + b^3 f(k, y_0^{-1}) y_0^{-4}; \\ y_{2k} &= kb + kca^{-1} y_0^{-1} + (1 + b^2 g_1(k)) y_0 + b g_2(k) y_0^2 + \\ &+ (-kca^{-1} + b^3 g_3(k)) y_0^3 + b^3 g_4(k, y_0) y_0^4, \end{aligned}$$

where

$$f_i(u), g_j(u) \in \mathbb{Z}[u],\ f(u,v), g(u,v) \in \mathbb{Z}[u,v], \deg f_i(u), \deg g_i(u) \le \ell\, (i,j = 1,2,\ldots)$$

The polynomials $f(u,v)$, $g(u,v)$ have degree at most ℓ in each variable. It enables us to prove the following assertions

Theorem 1. *The following estimates*

$$\sum_{\omega\in R_n^*} e_{p^n}(h_1\omega_k + h_2\omega_\ell) \le \begin{cases} 2p^{\frac{n+s}{2}}, & \text{if } gcd\left(h_1+h_2, h_1\left[\frac{k}{2}\right]+h_2\left[\frac{\ell}{2}\right], p^n\right) = p^s, \\ & s < n, \text{ and } k,\ell \text{ are integers of identical parity;} \\ 2^{\frac{n+1}{2}} p^{\frac{n+t}{2}}, & \text{if } gcd\,(h_1,h_2,p^n) = p^t,\ s < n, \text{ and} \\ & k,\ell \text{ are integers of different parity;} \end{cases}$$

hold.

Consider the sequence of points $\{Y_k\}$, where $Y_k = (y_k, y_{k+1}, \ldots, y_{k+r-1})$ and y_k is defined by the recursion (2). Let $D_N^{(3)}$ denote the discrepancy of points $\{\frac{Y_k^{(3)}}{p^n}\} \in [0,1)^3$, $k = 0,1,2,\ldots,N-1$.

Theorem 2. *Let the sequence $\{y_k\}$ be generated by (2) and have a maximal period $\tau = 2p^{n-\nu}$. Then we have*

$$D_\tau^{(3)} \le \frac{1}{p^n} + \frac{1}{p^{\frac{n-2\nu}{2}}}\left(1 + \frac{1}{p^\nu}\left(\frac{2}{\pi} logp^n + \frac{7}{5}\right)^3\right)$$

These results can be considered as the generalizations of appropriate assertions from [2]-[4].

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Products of normal subgroups of finite groups and n-multiply ω-saturated formations

A. F. Vasil'ev*, D. N. Simonenko♯

All groups considered are finite. In 1972 Bryce and Cossey proved the following famous result.

Theorem. *(Bryce and Cossey [1]). A subgroup-closed Fitting formation of soluble groups is totally saturated.*

Let ω be a set of primes. Recall [2], that a formation $\mathfrak{F}$ is said to be ω-saturated if the condition: $G/L \in \mathfrak{F}$ for a normal subgroup $L \subseteq \Phi(G) \cap O_\omega(G)$ always implies that $G \in \mathfrak{F}$.

In [3] A.N.Skiba proposed the concept of n-multiply saturated formations.

Every formation is 0-multiply ω-saturated. A saturated formation $\mathfrak{F}$ is called n-multiply ω-saturated $(n \geq 1)$ if $\mathfrak{F} = LF(f)$ and for every prime p formation $f(p)$ is either empty or $(n-1)$-multiply ω-saturated. A non-empty formation $\mathfrak{F}$ is called totally ω-saturated if it is n-multiply ω-saturated for every natural number.

In this talk a problem: by analogy with the theorem of Bryce and Cossey to characterize n-multiple ω-saturated formations in terms of the certain products of normal subgroups is considered.

Definition. *Let $\mathfrak{X}$ be a class of groups. A class of groups $\mathfrak{F}$ is called $D_{\mathfrak{X}}$-closed if every group $G = MN$ where M and N are normal $\mathfrak{F}$-subgroups of G and $M \cap N \in \mathfrak{X}$ belongs to $\mathfrak{F}$.*

Theorem. *Let ω be a set of primes. Let $\mathfrak{F}$ be a subgroup-closed formation of soluble groups and $\mathfrak{X} = \mathfrak{N}_\omega^n$, where n is positive integer. The following statements are equivalent.*

1) $\mathfrak{F}$ is $D_{\mathfrak{X}}$-closed formation;

2) $\mathfrak{F}$ is a n-multiply ω-saturated formation.

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Projectors of products of π-decomposable groups and Schunck π-classes

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All groups considered are finite. We use notations and definitions from [1], [2]. Let a group $G = AB$ be a product of subgroups A and B. In order to determine the structure of these groups it is of interest to know which subgroups of G are conjugate to a subgroup that inherits the factorization. A subgroup H of a group $G = AB$ is called factorized [3] if $H = (A \cap H)(B \cap H)$ and $A \cap B \leq H$. In [4-5] factorized subgroups of group $G = AB$ with nilpotent subgroups A and B were studied.

Let π be a set of primes and π' the complement of π in the set of all primes. A group G is called π-decomposable if $G = G_\pi \times G_{\pi'}$ and Hall π-subgroup G_π is nilpotent. A non-empty homomorph $\mathfrak{X}$ is a Schunck class if any group G, all of whose primitive factor groups are in $\mathfrak{X}$, is itself in $\mathfrak{X}$. A class $\mathfrak{X}$ of groups is called a π-class if $G/O_{\pi'}(G) \in \mathfrak{X}$ implies $G \in \mathfrak{X}$.

Theorem. *Let $\mathfrak{X}$ be a Schunck π-class. If a π-soluble group $G = AB$ is a product of two π-decomposable subgroups A and B and $\pi(A) \cap \pi(B) \subseteq Char(\mathfrak{X})$, then G has a unique factorized $\mathfrak{X}$-projector.*

Corollary 1. *Let $\mathfrak{F}$ be a saturated formation and π-class. If a π-soluble group $G = AB$ is a product of two π-decomposable subgroups A and B and $\pi(A) \cap \pi(B) \subseteq Char(\mathfrak{X})$, then G has a unique factorized $\mathfrak{F}$-projector.*

A π-Carter subgroup of a group G [6] is self-normalizing π-nilpotent subgroup H with $|H|_{\pi'} = |G|_{\pi'}$. A π-Gaschütz subgroup of group G [6] is π-supersoluble subgroup H such that $|H|_{\pi'} = |G|_{\pi'}$ and for all M, N, such that $H \leq N \leq M \leq G$ the index $|M : N|$ is not a prime.

Corollary 2. *Let a π-soluble group $G = AB$ be a product of two π-decomposable subgroups A and B. Then G has a unique factorized π-Carter subgroup.*

Corollary 3. *Let a π-soluble group $G = AB$ be a product of two π-decomposable subgroups A and B. Then G has a unique factorized π-Gaschütz subgroup.*

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Geometry of $\widehat{\varphi}$-representation for real numbers

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Let $\mathcal{F} = \{(c_n) : \ c_1, c_2 \in R, \ c_n = c_{n-1} + c_{n-2}, n \geq 3\}$ be a set of real Fibonacci sequences. Then the set $\langle \mathcal{F}, +, \cdot \rangle$ with the addition and the multiplication by a scalar is a bidimensional linear space.

Let $\mathcal{F}^1 = \left\{ (a_n) : \ a_1, a_2 \in R, \ a_n = a_{n-1} + a_{n-2}, n \geq 3, \sum\limits_{n=1}^{\infty} a_n < \infty \right\}$ be a set of convergent Fibonacci sequences.

Then $\left\langle \mathcal{F}^1, +, \cdot \right\rangle$ is one-dimensional linear subspace of the space $\mathcal{F}$. Every element of this subspace can be represented in the form:

$$\overrightarrow{a} = \left(a; \ a\widehat{\varphi}; \ a\widehat{\varphi}^2; \ \ldots; \ a\widehat{\varphi}^n; \ \ldots\right),$$

where $a \in R$, $\widehat{\varphi} = \frac{1-\sqrt{5}}{2}$ is a negative solution of the equation $x^2 - x - 1 = 0$.

Let $a = 1$. Then let us consider the cylindrical representation of real numbers with the help of series $\sum\limits_{n=0}^{\infty} \widehat{\varphi}^n$.

Theorem 1. *Any real $x \in \left[-1, -\widehat{\varphi}^{\ -1}\right]$ can be represented in the form:*

$$x = \sum_{n=0}^{\infty} \varepsilon_n(x) \widehat{\varphi}^{\ n} = \varepsilon_0(x)\widehat{\varphi}^0 + \varepsilon_1(x)\widehat{\varphi}^1 + \cdots + \varepsilon_n(x)\widehat{\varphi}^n + \cdots, \tag{1}$$

where $\varepsilon_n(x) \in \{0, 1\}, \ n \in N_0$.

A representation of $x \in \left[-1, -\widehat{\varphi}^{\ -1}\right]$ in form (1) is called *$\widehat{\varphi}$-expansion* of this number. We denote it briefly by $x = \Delta_{\varepsilon_0(x)\varepsilon_1(x)\ldots\varepsilon_n(x)\ldots}$ call *$\widehat{\varphi}$-representation* of the number x.

Let $(c_1c_2\ldots c_k)$ be a fixed set of figures of set $\{0,1\}$. A set

$$\Delta_{c_1\ldots c_k} = \left\{ x : x = \sum_{n=1}^{k} c_n\widehat{\varphi}^{n-1} + \sum_{m=k+1}^{+\infty} \varepsilon_m\widehat{\varphi}^{m-1}, \quad \varepsilon_m \in \{0,1\} \right\}$$

is called *the cylindrical set of rank k* with the base $(c_1c_2\ldots c_k)$.

In the talk the results of explorations of the properties and geometry of cylindrical sets are proposed. The task about a number of representations of the real number in the form (1) is examined.

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Semifields with distributive lattice of congruencies

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In this abstract we consider distributive semifields. *Semifield* is an algebraic structure, which is a commutative group with respect to addition and group with respect to multiplication, and multiplication are distributive with respect to addition from both sides. Class of unit 1 of any congruence of semifield is called its *kernel*. Set Con U of all kernels (of congruencies) of semifield U is a modular algebraic lattice with respect to inclusion. Semifield U is called *distributive* (*prime*) if lattice Con U is distributive (two-element). Additively idempotent semifields coincide with lattice-ordered groups; consequently, they are distributive.

Proper kernel P of semifield U is called *irreducible* if $A \cap B \subseteq P$ involves $A \subseteq P$ or $B \subseteq P$ for any $A,\ B \in$ Con U. Semifield U is called *biregular* (*Boolean*) if its arbitrary principal kernel (a) (its arbitrary kernel) has a complement in Con U. Semifield U is called *Gelfand* if for each of its different maximal kernels A and B there exist elements $a \in A \setminus B$ and $b \in B \setminus A$ such that $(a) \cap (b) = \{1\}$.

Notice that maximal kernels of any semifield are irreducible, biregular semifields are distributive and Gelfand, and fact that semifield is distributive is equivalent to the fact that all its proper kernels are intersections of irreducible kernels [1].

Considering semifields, we may use functional (sheaf) approach [1], with its help we may get following results.

Theorem 1. *Semifield is biregular if and only if it is Gelfand, and all its proper kernels are intersections of maximal kernels.*

Theorem 2. *For arbitrary semifield U following conditions are equivalent:*
(1) U is isomorphic to direct product of a finite number of prime semifields;
(2) U is Boolean, and $U = (a)$ for same $a \in U$;
(3) set Con U is finite, and any proper kernel in U is an intersection of maximal kernels.

Notice that if set Con U is finite then semifield U is distributive [2].

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Piltz's divisor problem in the matrix ring $M_2(\mathbb{Z})$

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Let $M_k(\mathbb{Z})$ denotes the ring of integer matrices of order k, $GL_k(\mathbb{Z})$ is the unite group of $M_k(\mathbb{Z})$. We denote the number of different (to association) representations of matrix $C \in M_k(\mathbb{Z})$ in the form $C = A_1A_2A_3,\ A_1, A_2, A_3 \in M_k(\mathbb{Z})$ as $\tau_k^{111}(C)$. It is interesting to investigate asymptotic behavior of the sum

$$T_k^{111}(x) =: \sum_{n\le x} \ \sum_{G\in M_k(\mathbb{Z}),\ |det\, G|=n}{}' \tau_k^{111}(G)$$

for different k (the second sum is taken throughout all matrices G accurate to integer unimodular factor).

G.Bhowmik and H.Menzer [1], H.-Q.Liu [2], A.Ivič [3], N.Fugelo and I.Velichko [4] studied an asymptotic behavior of the similar sums, but only for two factors.

Our aim is to construct the asymptotic formula for the function $T_2^{111}(x)$.

Lemma. *For any prime number $p > p_0,\ n \in \mathbb{N}$*

$$t_2^{111}(p^n) := \sum_{\substack{G\in M_2(\mathbb{Z})\\ |det\, G|=p^n}} \tau_2^{111}(G) = \begin{cases} (n+1)(n+3)p^n(1+O(1/p)), & \text{if } n = 2k, \\ \dfrac{(n+1)\,(n+2)}{2}p^n(1+O(1/p)), & \text{otherwise.} \end{cases}$$

Corollary.

$$F(s) := \sum_1^\infty \frac{t_2^{111}(n)}{n^s} = \frac{\zeta^3(s)\zeta^3(2s-1)}{\zeta(3s-1)}G(s), \tag{1}$$

where G(s) is absolutely convergent for $Re\, s > 4/7$.

Using (1) and the estimates of the fourth and sixth moments of $\zeta(s)$, the following result has been obtained :

Theorem. *For $x \to \infty$ and $\varepsilon > 0$ the estimate*

$$T_2^{111}(x) = xP_5(\log x) + O(x^{69/88+\varepsilon})$$

holds, where $P_5(u)$ is a polynomial of degree 5.

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The Cauchy-Laurent integral and Laplas transformation for a formal power series

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It will be presented the Laplas transformation for a formal power series in Bourel form $\oint\limits_{|\zeta|=\varepsilon} f(\zeta)e^{-s\zeta}d\zeta$, where contour integral is specially defined for a formal power series. Then will be considered an application of Laplas transformation to solving Volterra convolution integral equations.

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On a Cyclic Decomposition of S_n

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We denote by $S_k = Aut\{1,\dots,k\}$ and $C_k = <(1\dots k)>$ — subgroups of S_n, and $S_1 = C_1 = \{e\}$.

Lemma. *We have $C_n \cdot S_{n-1} = S_{n-1} \cdot C_n = S_n$ and any $\sigma \in S_n$ can be written in the form: $\sigma = c \cdot \tau = \widetilde{\tau} \cdot \widetilde{c}$ uniquely, where c, $\widetilde{c} \in C_n$, τ, $\widetilde{\tau} \in S_{n-1}$.*

Proof: if we have $c_1 \cdot \tau_1 = c_2 \cdot \tau_2$, then $c_2^{-1} \cdot c_1 = \tau_2 \cdot \tau_1^{-1} \in S_{n-1} \cap C_n = \{e\}$, because $S_{n-1} = St_{S_n}(n)$ and $St_{S_n}(n) \cap C_n \subseteq St_{C_n}(n) = \{e\}$.

Corollary 1. *The elements of S_{n-1} are the representatives of distinct residue classes (left or right) $S_n : C_n$, also the elements of C_n are the representatives of distinct residue classes (left or right) $S_n : S_{n-1}$.*

Corollary 2. $S_n = C_n \cdot C_{n-1} \cdot \ldots \cdot C_2 = C_2 \cdot C_3 \cdot \ldots \cdot C_n$ *and any $\sigma \in S_n$ can be written in the form:*

$$\sigma = (1\dots n)^{a_n}(1\dots n{-}1)^{a_{n-1}}\dots(12)^{a_2} = (12)^{b_2}(123)^{b_3}\dots(1\dots n)^{b_n}$$

uniquely, where $0 \le a_k$, $b_k \le k{-}1$.

Enumeration of S_n: if we associate $\phi\colon \sigma \to (a_n a_{n-1} \dots a_2)$, $0 \le a_k \le k{-}1$ and define the lexicographic order on the $\phi(S_n)$, we get the enumeration of permutations, which respects the embedding: $S_1 \subset S_2 \subset S_3 \subset \cdots \subset S_n$.

Example. *Let us enumerate in this way permutations of the set* $\{1,2,3,4\}$*:*

$$1234, 2134, 2314, 3214, 3124, 1324, 2341, 3241, 3421, 4321, 4231, 2431,$$

$$3412, 4312, 4132, 1432, 1342, 3142, 4123, 1423, 1243, 2143, 2413, 4213.$$

This algorithm in the beginning reminds the Knuth's procedure of the enumeration of permutations with the Sims's table [1, chap. 7.2.1.2], but with the differing result. Thus, if we change an order of the cyclic decomposition and a choice of the cycles generating C_k, we receive the different ways of the enumeration of S_n.

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Compound Inversive Congruential Generator with Prime Power Modulus

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Let $p_1, ..., p_r$ be different prime numbers, $p_i \geq 5, i = 1, ..., r$, and let $m_1, ..., m_r$ be natural numbers. Consider the pairs of numbers (a_i, b_i), such that $(a_i, p_i) = 1$, $b_i \equiv 0\ (mod\ p_i)$, $i = 1, ..., r$. Take fixed $y_0^{(i)} \in Z_{p_i^{m_i}}, (y_0^{(i)}, p_i) = 1$, and consider for each $i = 1, ..., r$ inversive congruential generator

$$G_i : \qquad y_{n+1}^{(i)} \equiv \frac{a_i}{y_n^{(i)}} + b_i (mod\ p_i), \qquad n = 0, 1, 2...$$

Generators G_i provide periodic sequence with period $\tau_i \leq 2p_i^{m_i - \nu_i}$, where ν_i is the power with which p_i appears in the canonical decomposition of b_i. It is known that if $a_i^2 \not\equiv (y_0^{(i)})^4 (mod\ p_i)$, then $\tau_i = 2p^{m_i - \nu_i}$.

Take fixed $c_i \in Z_{p_i^{m_i}}, (c_i, p_i) = 1$, and consider recursion

$$G_0 : \qquad z_{n+1}^{(i)} \equiv a_i c_i^2 (z_n^{(i)})^{-1} + c_i b_i (mod\ p_i^{m_i}), \qquad z_0^{(i)} \equiv c_i y_0^{(i)} (mod\ p_i^{m_i})$$

Then the sequence $\{x_n\}$, where

$$x_n \equiv x_n^{(1)} + ... + x_n^{(r)} (mod\ 1), n \geq 0,$$

where

$$x_n^{(i)} = \frac{z_n^{(i)}}{p_i^{m_i}}$$

will be called compound inversive sequence of pseudorandom numbers (PRN), and the generator G_0 — the compound inversive generator.

Generator G_0 is the generalization of compound inversive generator, it was first studied in works of J.Eichenauer[1, 2, 3].

The purpose of our research was the construction of top and bottom estimations of discrepancy function on periodic parts of two sequences of s-dimensional points.

$x_n = (x_{sn}, x_{sn+1}, ..., x_{sn+s-1}) \in [0,1)^s$("non-overlapping" points)

$\tilde{x}_n = (x_n, x_{n+1}, ..., x_{n+s-1}) \in [0,1)^s$("overlapping" points).

It was proved, that for "almost all" gangs $c_1, ..., c_r$ the following right inequations hold

$$(2\sqrt{s})^r N^{-\frac{1}{2}} \ll D_N^{(s)}(x_0, ..., x_{N-1}) \ll (2\sqrt{s})^r N^{-\frac{1}{2}} (\log \tau)^r,$$

where $\tau_i = 2p_1^{m_1-\nu_1}...p_r^{m_r-\nu_r}$, $N \gg \tau^{\frac{1}{2}}$,

$$D_N^{(s)}(x_0, ..., x_{N-1}) := \sup_{\Delta \subset [0,1)^s} \left| \frac{A_N(\Delta)}{N} - \mid \Delta \mid \right|$$

$A_N(\Delta)$ – number of points of sequence $x_0, x_1, ..., x_{N-1}$, that got into parallelepiped Δ; $\mid \Delta \mid$ – cubage of Δ , and sup is taken over all Δ from $[0,1)^s$.

Similar estimations are fair also for points $\tilde{x}_n$, if $s < min(p_1, ..., p_r)$.

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Vector Bundles over Noncommutative Nodal Curves

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We investigate certain noncommutative projective curves such that all their singularities are nodal algebras [1]. They generalize projective configurations considered in the paper [2]. We prove that these noncommutative nodal curves are vector bundles tame and describe vector bundles over them. The proof uses the technique of "matrix problems", which was also used in [2], or, more exactly, "representations of bunches of semi-chains" [3].

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Random walk on finite 2-transitive group generated by its natural character

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Let P be a probability on a finite group G, $U(g) = \frac{1}{|G|}$ the uniform probability on G, $P^{(n)} = P * \ldots * P$ (n times) an n-fold convolution of P. Under well known conditions $P^{(n)} \to U$ if $n \to \infty$ (see e.g. [1]). A lot of estimates for the rate of that convergence are found in different norms ([2]). We give exact formula for some groups in the norm $\|F\| = \sum_{g \in G} |F(g)|$, where F is a function on group G.

Let G be 2-transitive permutation group of degree d, χ the natural permutation character of G. The function $P(g) = \frac{1}{|G|} \sum_{g \in G} \chi(g)$ is a probability on the group G.

Theorem 1.

$$\|P^{(n)} - U\| = \frac{2R}{(d-1)^{n-1}|G|},$$

where R is the number of regular (i.e. without fixed points) permutations in G.

Let

$$E_d = \sum_{r=0}^{d} \frac{(-1)^r}{r!}.$$

Corollary 1. *1) If G is a symmetric group of degree d, then*

$$\|P^{(n)} - U\| = \frac{2E_d}{(d-1)^{n-1}}.$$

2) If G is an alternating group of degree d, then

$$\|P^{(n)} - U\| = \frac{2(E_{d-2} + \frac{2}{d!}(-1)^{d-1}(d-1))}{(d-1)^{n-1}}.$$

3) If G is a Zassehaus group, then

$$\|P^{(n)} - U\| = \frac{|G| - (d-1)(d-2)}{(d-1)^{n-1}|G|}.$$

A Zassehaus group is a 2-transitive group in which stabilizer of any three distinct points is trivial.

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On minimal subgroups of finite groups

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All groups considered are finite.

Let G be a group. A minimal subgroup of G is a subgroup of prime order. For a group of even order, it is also helpful to consider cyclic subgroups of order 4. There has been a considerable interest in studying the group structure under the assumption that minimal subgroups and cyclic subgroups of order 4 are well-situated in G.

Definition 1. *(see [1]). (1) An element x of a group G is called Q-central if there exists a central chief factor A/B of G such that $x \in A \setminus B$.*

(2) An element x of a group G is called $Q\mathfrak{U}$-central if there exists a cyclic chief factor A/B of G such that $x \in A \setminus B$.

Definition 2. *(see [2]). An element $x \in G$ is called a Q_8-element if there exists a section A/B of G such that $xB \in A/B$, $A/B \simeq Q_8$ (the quaternion group of order 8) and the order of x is equal to the order of xB in A/B.*

Theorem 1. *Let G be a finite group, and H a normal subgroup of G. If all elements of prime order in H and all Q_8-elements of order 4 in H are $Q\mathfrak{U}$-central in G, then every G-chief factor of H is cyclic.*

Theorem 2. *Let G be a finite group, and H be a normal subgroup of G. If all elements of prime order in H and all Q_8-elements of order 4 in H are Q-central in G, then H is contained in the hypercenter of G.*

Theorem 3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Suppose that every Q_8-element of order 4 in $F^*(G^{\mathfrak{F}})$ is Q-central in G. If $G \notin \mathfrak{F}$, then there exists an element of prime order in $F^*(G^{\mathfrak{F}}) \setminus Z_{\mathfrak{F}}(G)$.*

Here $Z_{\mathfrak{F}}(G)$ is the $\mathfrak{F}$-hypercenter of G.

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On the divisibility of partial isotone transformations

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Denote by $T(X)$ the semigroup of all transformations $\alpha : X \to X$ of a set X. If X is a partially ordered set then a mapping $\alpha : X \to X$ is called *isotone* when $\forall x, y \in X\ x \leqslant y \ \Rightarrow\ x\alpha \leqslant y\alpha$. The set of all isotone transformations $\alpha : X \to X$ we denote by $O(X)$.

Let S be a semigroup and $a, b \in S$. We say that a is left divided by b (and we write $a \leqslant_l b$) if $a \in S^1 b$. The Green's relation $\mathcal{L}$ can be defined as follows: $\mathcal{L} = \leqslant_l \cap \geqslant_l$. The relations $\leqslant_r$, $\mathcal{R}$ are defined dually.

An arbitrary linearly ordered set is called a *chain*.

The structure of Green's relations on the semigroup $T(X)$ is well known. Namely, (i) if $\alpha, \beta \in T(X)$ then $\alpha\mathcal{R}\beta \Leftrightarrow \ker\alpha = \ker\beta$; (ii) if $\alpha, \beta \in T(X)$ then $\alpha\mathcal{L}\beta \Leftrightarrow \operatorname{im}\alpha = \operatorname{im}\beta$.

If we take $O(X)$ instead of $T(X)$ then propositions (i), (ii) will be wrong in general. However the following propositions are right ([1]):

Proposition 1. *(i) If X is an arbitrary chain and $\alpha, \beta \in O(X)$ then $\alpha \leqslant_l \beta \Leftrightarrow \operatorname{im}\alpha \subseteq \operatorname{im}\beta$; (ii) if X is a finite chain and $\alpha, \beta \in O(X)$ then $\alpha \leqslant_r \beta \Leftrightarrow \ker\alpha \supseteq \ker\beta$.*

Corollary 1. *(i) If X is an arbitrary chain and $\alpha, \beta \in O(X)$ then $\alpha\mathcal{L}\beta \Leftrightarrow \operatorname{im}\alpha = \operatorname{im}\beta$; (ii) if X is a finite chain and $\alpha, \beta \in O(X)$ then $\alpha\mathcal{R}\beta \Leftrightarrow \ker\alpha = \ker\beta$.*

Let $P(X, Y)$ be the set of all partial mappings $\alpha : X \to Y$, i.e., the mappings which are defined maybe for not all $x \in X$. If $X = Y$ we write $P(X)$ instead of $P(X, Y)$. The *domain* $\operatorname{dom}\alpha$ of $\alpha \in P(X, Y)$ is the set $\{x \mid \exists y : y = x\alpha\}$, and the *image* $\operatorname{im}\alpha$ is the set $\{y \mid \exists x : y = x\alpha\}$. The *kernel* of α is the set $\ker\alpha = \{(x, y) \mid x\alpha = y\alpha\}$. It is an equivalence relation on the set $\operatorname{dom}\alpha$ (but not on the set X, generally).

Let $(X, \leqslant)$ and $(Y, \leqslant')$ be the partially ordered sets. A mapping $\alpha \in P(X, Y)$ is called *isotone* if $\forall x, y \in \operatorname{dom}\alpha$ $(x \leqslant y \Rightarrow x\alpha \leqslant' y\alpha)$.

The set of all isotone transformation $\alpha \in P(X, Y)$ we denote by $PO(X, Y)$.

Proposition 2. *Let Γ_1, Γ_2 and Γ_3 be chains, $\alpha \in PO(\Gamma_1, \Gamma_2)$ and $\beta \in PO(\Gamma_3, \Gamma_2)$. Then $(\exists\gamma \in PO(\Gamma_1, \Gamma_3)\ (\alpha = \gamma\beta)) \Leftrightarrow (\operatorname{im}\alpha \subseteq \operatorname{im}\beta)$.*

Corollary 2. *(i) If X is an arbitrary chain and $\alpha, \beta \in PO(X)$ then $\alpha \leqslant_l \beta \Leftrightarrow \operatorname{im}\alpha \subseteq \operatorname{im}\beta$; (ii) if X is an arbitrary chain and $\alpha, \beta \in PO(X)$ then $\alpha\mathcal{L}\beta \Leftrightarrow \operatorname{im}\alpha = \operatorname{im}\beta$.*

The dual proposition of Corollary 2(i) is wrong. The dual proposition of Corollary 2(ii) has the following form.

Proposition 3. *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be chains, $\alpha \in PO(\Gamma_1, \Gamma_2)$, $\beta \in PO(\Gamma_1, \Gamma_3)$. There are the transformations $\delta_1 \in PO(\Gamma_2, \Gamma_3)$ and $\delta_2 \in PO(\Gamma_3, \Gamma_2)$ such that $\alpha\delta_1 = \beta$, $\beta\delta_2 = \alpha$ exist if and only if $\ker\alpha = \ker\beta$.*

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A commutative Bezout ring of stable range 2 with right (left) Krull dimension is an elementary divisor ring

B. Zabavsky

Throughout this notes R is assumed to be a commutative ring with $1 \neq 0$. Following Kaplansky [1] if for every matrix A over a ring R there exist invertible matrices P, Q such that PAQ is a diagonal matrix (d_{ij}) with the property that every d_{ij} is a divisor of $d_{i+1,i+1}$, then R is an elementary divisor ring. If every 1 by 2 matrix over R admits diagonal reduction then R is an Hermite ring [1]. By a Bezout ring we mean a ring in which all finitely generated ideals are principal. A ring R is a ring of stable range 2 if for every $a, b, c \in R$ such that $aR + bR + cR = R$ there exist $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$ [2].

Note that a Bezout ring is a Hermite ring if and only if the stable range of R is equal to 2 [2].

Theorem 1. *Let R be a Bezout ring of stable range 2 with right (left) Krull dimension. Then R is an elementary divisor ring.*

Theorem 2. *Let R be a Bezout ring of stable range 2 with Noetherian spectrum. Then R is an elementary divisor ring.*

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On the distribution of inverse congruential pseudorandom Gaussian numbers

S. Zadorozhny

Let p be a prime Gaussian, $N(p)$ denote the norm of p. Let α and β belong to the ring of Gaussian integers, $(\alpha, p) = 1$, $\beta = 0 \pmod p$. We define the following recursion

$$w_{n+1} = \alpha w_n^{-1} + \beta \pmod{p^m} \tag{1}$$

Where $m \in \mathbb{N}$, $w_0 \in \mathbb{Z}[i]$, $(w_0, p) = 1$ are fixed, w^{-1} denotes a multiplicative inverse of $w \mod p^m$

The sequence $\{w_n\}$ generated by (1) is called the sequence of inverse congruential pseudorandom Gaussian numbers. This definition is an analog of the sequence of pseudorandom numbers over $\mathbb{Z}$ which was first introduced by Niederreiter and Shparlinski [1].

We obtain the estimate of special exponential sum

$$\sigma_r(p^m) = \sum_{\substack{w_0 \in \mathbb{Z}_{p^m} \\ (w_0, p) = 1}} e^{\pi i Sp(\frac{w_r - w_0}{p^m})}$$

where $Sp(z) = 2Rez$, $z \in \mathbb{C}$.

Theorem 1. *For $m \geq 2$ and odd r we have*

$$\sigma_r(p^m) << N(p)^{\frac{m}{2}}.$$

Theorem 2. *Let $\beta = \beta_0 p^b$, $(\beta_0, p) = 1$ and r is even. We have*

$$\sigma_r(p^m) \leq DN(p)^{\frac{m+b+\varepsilon_b}{2}}, \quad \varepsilon_b = \begin{cases} 0 & if \quad m = b \pmod 2, \\ 1 & if \quad m \neq b \pmod 2, \end{cases}$$

where D is the number of solutions of the congruence

$$2\beta_0 u^3 = (-1)^{r/2} \pmod{p^{m_1}}, \qquad m_1 = [(m-b)/2].$$

These results allow to obtain a non-trivial estimate for the discrepancy of the sequence $\{\frac{w_n}{N(p)^m}\}$ in the square $[-1, 1]^2$.

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On Local Fischer Classes

E. N. Zalesskaya[♮], S. N. Vorob'ev[♮]

We use standard notation and terminology taken from [1]. All groups considered are finite.

The idea of localization is a leading one in the theory of groups. In this paper we deal with the local methods in the theory of Fischer classes which are analogous to the local methods in the theory of normal Fitting classes.

We remind that a Fitting class $\mathfrak{F}$ is called a Fischer class if from $G \in \mathfrak{F}$, $H \subseteq K \subseteq G$, $K \lhd G$ and $H/K \in \mathfrak{N}$ it always follows that $H \in \mathfrak{F}$ where $\mathfrak{N}$ is the class of all nilpotent groups.

Definition. *Let $\mathfrak{Y}$ be a subgroup-closed formation. A Fitting class $\mathfrak{F}$ is called a Fischer $\mathfrak{Y}$-class if from $G \in \mathfrak{F}$, $H \subseteq K \subseteq G$, $K \lhd G$ and $H/K \in \mathfrak{Y}$ it always follows that $H \in \mathfrak{Y}$.*

It is clear that Fischer $\mathfrak{Y}$-class is a Fischer class in the case $\mathfrak{Y} = \mathfrak{N}$.

Let us define the operator $S_{F\mathfrak{Y}}$ on the class of groups $\mathfrak{X}$ in the following way:

$$S_{F\mathfrak{Y}}\mathfrak{X} = (G : H \subseteq G \in \mathfrak{X} \text{ and } H^{\mathfrak{Y}} \lhd \lhd G).$$

It is easy to see that the operator $S_{F\mathfrak{Y}}$ is a closure operator, i.e. for any classes of groups $\mathfrak{X}$ and $\mathfrak{Z}$ it is true that:

(1) $\mathfrak{X} \subseteq S_{F\mathfrak{Y}}\mathfrak{X}$;

(2) from $\mathfrak{X} \subseteq \mathfrak{Z}$ it follows that $S_{F\mathfrak{Y}}\mathfrak{X} \subseteq S_{F\mathfrak{Y}}\mathfrak{Z}$;

(3) $S_{F\mathfrak{Y}}\mathfrak{X} = S_{F\mathfrak{Y}}(S_{F\mathfrak{Y}}\mathfrak{X})$.

We have proved the following theorem.

Theorem. *A Fitting class $\mathfrak{F}$ is a Fischer $\mathfrak{Y}$-class if and only if $\mathfrak{F}$ is an $S_{F\mathfrak{Y}}$-closed class.*

If $\mathfrak{Y} = \mathfrak{N}$ we obtain the well-known result of Hawkes [2].

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Abelian groups of triangular matrices

R. Zatorsky

Two abelian groups of triangular matrices were constructed using sets $\Xi(n)$ and parafunction of triangular matrices.

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On the algebraic curves over pseudoglobal fields

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According to [1] by a *pseudoglobal* field we mean an algebraic function field in one variable over a pseudofinite constant field. Recall [2] that a field k is called *pseudofinite*, if it is perfect, has unique extension of degree n for every positive integer n, and each nonempty absolutely irreducible variety over k has a k-rational point.

Let X_K be a connected smooth absolutely irreducible curve over K. For a valuation v denote by K_v the corresponding completion of K, $\mathrm{Br}\,(X)$ denotes the Brauer group of X, and Ш(A) is the Tate-Shafarevich group of the Jacobian A of X.

M. Artin and J. Milne [3] showed that these two groups are closely related. Subsequently, the study of relationships between them was continued by C.D. Gonzalez-Avilez [4]. The purpose of this announcement is to investigate similar relationships for curves over a pseudoglobal field.

Theorem 1. *Let X be a connected smooth absolutely irreducible curve over a pseudoglobal field K of characteristic zero. Suppose that X has index δ and period δ'. Then there exists an exact sequence*

$$0 \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \mathrm{Br}\,(X)' \longrightarrow \text{Ш}(\mathrm{Pic}\,(X)) \longrightarrow \mathbb{Q}/\Delta^{-1}\mathbb{Z},$$

where T_0 and T_1 are finite groups of orders $(\delta/\delta')\cdot[A(K):\mathrm{Pic}^0(X_K)]$ and $\prod\delta_v/\Delta$ respectively, and Δ is the least common multiple of local indices δ_v of X.

Corollary. *Suppose that all local indices δ_v of X are equal to 1. Then: i) The index of X equals 1; ii)* $\mathrm{Br}\,(X) =$ *Ш(A).*

The proof is based on the Hasse principle for the Brauer groups of K, the Tate-Shafarevich duality for abelian varieties over pseudolocal fields as well as on the arguments used in the case of a global ground field, see [3], [4]. We also extend Lichtenbaum's theorem [5] to the case of curves over pseudoglobal fields.

Theorem 2. *Let X be a curve of genus g over a pseudolocal field K, and let δ' and δ be period and index of X. Then: a) $\delta'|(g-1)$; b) $\delta=\delta'$ or $2\delta'$; and c) If $\delta = 2\delta'$, then $(g-1)/P$ is odd.*

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Hypercritical graphical quadratic forms and minimal tournaments with non-sign definitely Tits forms

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In the classic monograph of C. Ringel [1] there was introduced the concept of a graphical integral quadratic form, which is a generalization of the notion of the Tits form of partially ordered set, and were described the all critical (i.e. minimal non-weakly positive) graphical integral quadratic forms in the shape of an explicit list of "dotted" graphs (i.e. undirected graphs without loops and multiple edges, the all ones of which are dotted) corresponding to them. In presented report, this result is generalized by the author at the case of hypercritical (i.e. minimal non-weakly non-negative) graphical integral quadratic forms.

On the other hand, in the work of the author [2] there was introduced the concept of the Tits quadratic form of tournament i.e. anti-symmetric reflexive relation on a finite set (which naturally generalizes the notion of the Tits quadratic form of partially ordered set) as well as there was introduced the concept of $\mathbf{(0,1)}$ -equivalence of (dotted) graphs (this equivalence arise from a certain special integer linear transformation of corresponding graphical integral quadratic forms). At the same time, in [2] it was proved that the minimal tournaments (in particular, partially ordered sets) Tits form of which is not positive /respectively, non-negative/ definitely (called by the author as weakly critical / respectively, weakly hypercritical/ tournaments) are exactly those, underlined (dotted) graphs of which are $\mathbf{(0,1)}$ -equivalent to (dotted) graphs of critical /respectively, hypercritical/ graphical integral quadratic forms (or, that is the same, Tits quadratic forms of such tournaments must be $\mathbf{(0,1)}$ -equivalent to some critical /respectively, hypercritical/ graphical integral quadratic forms).

The obtained results allow to give an explicit description of the all weakly critical and, respectively, weakly hypercritical tournaments (in particular, of corresponding partially ordered sets).

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Of the connection of involutions in the rings of matrices

V. Zelisko†, M. Kuchma‡

Let K be a commutative principal ideal domain with involution ∇ [1].

In [2] involution ∇ is defined on the ring of matrices $M_n(K)$ as:

$$A^{\nabla} = (a_{ij})^{\nabla} = (a_{ji}^{\nabla}). \tag{1}$$

In [3] for matrices $A \in M_{2^n}(K)$ of a form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

a simplectic involution $*$ is defined as:

$$A^* = \begin{pmatrix} {A_4}^* & -{A_2}^* \\ -{A_3}^* & {A_1}^* \end{pmatrix}, \tag{2}$$

where ${A_i}^*$ is again defined by (2) in the ring $M_{2^{n-1}}(K)$.

In a ring $M_{2n}(K)$ we can define a mixed involution as:

$$A^{\#} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}^{\#} = \begin{pmatrix} {A_4}^{\nabla} & -{A_2}^{\nabla} \\ -{A_3}^{\nabla} & {A_1}^{\nabla} \end{pmatrix},$$

where ${A_i}^{\nabla}$ are defined by (1).

A matrix $A \in M_n(K)$ is called $\nabla-$ symmetric if $A^{\nabla} = A$. Similarly, $A \in M_{2^n}(K)$ is called $*-$ symmetric if $A^* = A$, and $A \in M_{2n}(K)$ is $\#-$symmetric if $A^{\#} = A$.

Theorem 1. *For any matrix $A \in M_{2^n}(K)$ with involution $*$ and for matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ holds*

$$A^* = (J^{\otimes n})^t A^t J^{\otimes n},$$

where $J^{\otimes n} = \underbrace{(J \otimes \ldots \otimes (J \otimes J))}_{n}$ and $J^{\otimes n} = = \pm (J^{\otimes n})^t$.

In particular, if $n = 2k+1$, then $J^{\otimes n} = -(J^{\otimes n})^t$, if $n = 2k$, then $J^{\otimes n} = (J^{\otimes n})^t$.

Theorem 2. *For any matrix $A \in M_{2^n}(K)$ with involutions $*$, $\#$ and transpose involution ∇ $(\nabla = t)$ holds*

$$A^* = (J^{\otimes n})^{\nabla} A^{\#} J^{\otimes n}.$$

Theorem 3. *A direct product of $\nabla-$ symmetric matrices is $\#-$ symmetric matrix.*

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Socle sequences of abelian regular semiartinian rings

J. Žemlička

A module M is said to be *semiartinian* if it contains a smooth strictly increasing (so called Loewy) chain of submodules $(S_\alpha \mid \alpha \leq \sigma)$ such that $S_{\alpha+1}/S_\alpha = Soc(M/S_\alpha)$ and we say that a ring R is right semiartinian provided that the right module R_R is semiartinian. It is shown in [2, Théorème 3.1, Proposition 3.2] that structure questions about commutative semiartinian rings can be translate to those over commutative regular semiartinian rings, namely, a commutative ring R is semiartinian iff $R/J(R)$ is (commutative) regular and the Jacobson radical $J(R)$ is T-nilpotent (cf. [3, Lemma 3.5]).

Suppose that R is an *abelian regular* (i.e. every finitely generated right ideal is generated by some central idempotent) right semiartinian ring and denote by $(S_\alpha \mid \alpha \leq \sigma + 1)$ its Loewy chain. We investigate a family of cardinalities $\lambda_\alpha = gen(S_{\alpha+1}/S_\alpha)$ which forms a natural invariant of such rings. Since $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} K_{\alpha\beta}$, where simples $K_{\alpha\beta}$ have natural structure of skew-fields, there exist injective ring-homomorphisms $\varphi_\alpha : R/S_\alpha \to \prod_{\beta<\lambda_\alpha} K_{\alpha\beta}$. As a consequence of this fact we can formulate

Proposition. *If $\alpha \leq \beta \leq \sigma$, then $|\sigma| \leq 2^{\lambda_0}$ and $\lambda_\beta \leq 2^{\lambda_\alpha}$.*

Applying classical results of combinatorial set theory (including properties of partitions proved in [1]) we can enhance Proposition by

Theorem. *Suppose that the Generalized Continuum Hypothesis holds and α and δ are ordinals satisfying $\alpha + \delta \leq \sigma$. If $cf(\lambda_\alpha) > \max(|\delta|, \omega)$, then $\lambda_{\alpha+\delta} \leq \lambda_\alpha$. Otherwise $\lambda_{\alpha+\delta} \leq (\lambda_\alpha)^+$.*

On the other hand, we present constructions (based on tools introduced in [3]) of abelian regular semiartinian algebras with specified transfinite sequences $(\lambda_\alpha \mid \alpha \leq \sigma)$. Finally, several examples illustrate the fact that both our methods and results depend on a model of set theory.

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Canonical form of pairs of 4-by-4 Hermitian matrices under unitary similarity

Nadya Zharko

This is joint work with Vladimir Sergeichuk.

We give canonical matrices of pairs of self-adjoint operators in a four-dimensional unitary space. Their matrices are Hermitian, thus we obtain a canonical form of pairs of Hermitian matrices for unitary similarity.

Each square complex matrix M is uniquely represented in the form

$$A + Bi, \qquad A := \frac{M^* + M}{2}, \quad B := \frac{M^* - M}{2} i,$$

in which A and B are Hermitian, and so this classification problem could be reduced to the canonical form problem for 4×4 complex matrices up to unitary similarity, which can be solved by Littlewood algorithm [2]. However, the canonical form of (A, B) obtained in such a way is rather complicated. That is why it is preferred to consider these problems separately. Note that invariants of 4×4 matrices under orthogonal and unitary similarities and their applications are considered in [1].

Our list of canonical pairs of 4-by-4 Hermitian matrices under unitary similarity is too long to be put here, but we give an analogous result for 3-by-3 Hermitian matrices in the following theorem.

Theorem 1. *Each pair of 3-by-3 Hermitian matrices is unitary similar to exactly one pair from the following list:*

$$\left(\begin{bmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{bmatrix}, \begin{bmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{bmatrix}\right), \qquad \left(\begin{bmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \gamma\end{bmatrix}, \begin{bmatrix}\lambda & 0 & a\\ 0 & \lambda & 0\\ a & 0 & \nu\end{bmatrix}\right),$$
$$\left(\begin{bmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \gamma\end{bmatrix}, \begin{bmatrix}\lambda & 0 & a\\ 0 & \lambda' & b\\ a & b & \nu\end{bmatrix}\right), \qquad \left(\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{bmatrix}, \begin{bmatrix}\lambda & a & b\\ a & \mu & c\\ b & \bar{c} & \nu\end{bmatrix}\right),$$

in which α, β, γ are distinct real numbers, $\lambda, \lambda', \mu, \nu$ are real numbers, $\lambda \neq \lambda'$, a, b are nonnegative real numbers. In the last pair, c is a complex number and if $a = 0$ or $b = 0$ then c is a nonnegative real number.

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On commutative dimonoids

Anatolii V. Zhuchok

Jean-Louis Loday [1] introduced the notion of a dimonoid. A set D equipped with two associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$(x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$ is called a dimonoid. If the operations $\prec$ and $\succ$ coincide, then the dimonoid becomes a semigroup.

We introduce the notion of a diband of dimonoids. We call a dimonoid $(D, \prec, \succ)$ an idempotent dimonoid or a diband if $x \prec x = x = x \succ x$ for all $x \in D$. Let J be some idempotent dimonoid. We call a dimonoid $(D, \prec, \succ)$ a diband of subdimonoids D_i $(i \in J)$ if $D = \bigcup_{i \in J} D_i$, $D_i \bigcap D_j = \emptyset$ for $i \neq j$ and $D_i \prec D_j \subseteq D_{i \prec j}, D_i \succ D_j \subseteq D_{i \succ j}$ for all $i, j \in J$. If J is a band (=idempotent semigroup), then we say that $(D, \prec, \succ)$ is a band J of subdimonoids D_i $(i \in J)$. If J is a commutative band, then we say that $(D, \prec, \succ)$ is a semilattice J of subdimonoids D_i $(i \in J)$.

We call a dimonoid $(D, \prec, \succ)$ commutative if both semigroups $(D, \prec)$ and $(D, \succ)$ are commutative. We say that a commutative dimonoid $(D, \prec, \succ)$ is archimedean if both semigroups $(D, \prec)$ and $(D, \succ)$ are archimedean.

Theorem 1. *A semigroup $(D, \prec)$ of a commutative dimonoid $(D, \prec, \succ)$ is archimedean (respectively, regular) if and only if a semigroup $(D, \succ)$ of a commutative dimonoid $(D, \prec, \succ)$ is archimedean (respectively, regular). Every commutative dimonoid $(D, \prec, \succ)$ is a semilattice Y of archimedean subdimonoids D_i, $i \in Y$.*

This result is a generalization of a theorem by Tamura and Kimura [2] about the decomposition of commutative semigroups into semilattices of archimedean subsemigroups.

In addition, we construct a free commutative dimonoid and describe the least idempotent congruence of this dimonoid.

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J-, D- and H-cross-sections of symmetric inverse 0-category

Y. Zhuchok

Let $Map_b(A;B)$ be the set of all bijective mappings of A on a set B, and let $BSymX$ be the union of all sets $Map_b(A;B)$, where $A, B \subseteq X, A \neq \emptyset \neq B$. On the set $BSym^0X = BSymX \cup \{0\}$ we will define an operation $*$ by the rule: if $\varphi \neq 0 \neq \psi$ and $Im\varphi = Dom\psi$, then $\varphi * \psi = \varphi \circ \psi$, where $\circ$ is the ordinary composition of mappings, otherwise $\varphi * \psi = 0$. Under such operation, the set $BSym^0X$ is a semigroup which is called the symmetric inverse 0-category on the set X. Symmetric semigroups and their different properties have been studied by many authors (see e.g. 1-4). Here cross-sections of Green relations J, D and H on semigroup $BSym^0X$ are investigated.

Lemma 1. *Let $\varphi, \psi \in BSym^0X$. Then*
(i) $(\varphi;\psi) \in J \Leftrightarrow (\varphi;\psi) \in D \Leftrightarrow (\varphi = \psi = 0 \vee \varphi \neq 0 \neq \psi,\ |Im\varphi| = |Im\psi|)$;
(ii) $(\varphi;\psi) \in H \Leftrightarrow (\varphi = \psi = 0 \vee \varphi \neq 0 \neq \psi, Dom\varphi = Dom\psi, Im\varphi = Im\psi)$.

Let ρ be an equivalence relation on a semigroup S. A subsemigroup T of S is called a ρ-cross-section if T contains exactly one element from every equivalence class. For all $i \in I = \{1, 2, ..., n\}$ we will put $S_i^0 = \{\varphi \in BSymX \| Dom\varphi| = i\} \cup \{0\}, |X| = n$.

Lemma 2. *For every $i \in I$ subsets $\{0, \varphi\} \subseteq S_i^0$, where $\varphi^2 = \varphi$ or $\varphi^2 = 0$, and only these subsets are J-cross-sections of semigroup S_i^0.*

Denote by $q(\rho^S)$ the quantity of all ρ-cross-sections of the semigroup S and by C_k^l the quantity of all l-element subsets of a k-element set.

Theorem. *Let X be a n-element set. Then*

$$q(J^{BSym^0X}) = q(D^{BSym^0X}) = \prod_{i=1}^{n}(C_n^i + ((C_n^i)^2 - C_n^i)i!).$$

The quantity of all H-cross-sections of the symmetric inverse 0-category on finite set is also described.

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Alternative superalgebras on one odd generator

Natalia Zhukavets

For every variety of algebras $\mathcal{V}$, one can consider the corresponding $\mathcal{V}$-Grassmann algebra, which is isomorphic as a vector space to the subspace of all skew-symmetric elements of the free $\mathcal{V}$-algebra. It seems interesting to construct a base for this subspace. Due to [3, 6], the problem is reduced to the free $\mathcal{V}$-superalgebra on one odd generator, which is easier to deal with.

In [4] we constructed a base of the free alternative superalgebra $\mathcal{A}$ on one odd generator. As a corollary we obtained a base of the alternative Grassmann algebra. We also described the nucleus and the center of $\mathcal{A}$ and found a new element of minimal degree in the radical of the free alternative algebra.

The knowledge of a base of the free alternative superalgebra $\mathcal{A}$ on one odd generator permits to investigate the structure of skew-symmetric identities and central elements in any given alternative algebra. In [5] we classified all super-identities and central functions of the free quadratic alternative superalgebra on one odd generator. We also proved that in characteristic 0 the skew-symmetric identities and central functions of octonion algebras coincide with those for the class of all quadratic alternative algebras.

In case of alternative algebras, the Dubnov-Ivanov-Nagata-Higman theorem is not true in general, but the Zhevlakov theorem establishes that every alternative nil-algebra is solvable. In [2] we constructed bases of free alternative nil-superalgebras of indices 2 and 3 on one odd generator and computed their indices of solvability. We considered also the corresponding Grassmann algebra and showed that the well known Dorofeev's example [1] of solvable non-nilpotent alternative algebra is its homomorphic image.

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Rigid quivers

D. V. Zhuravlyov

In the paper [1] described some classes of finite partially ordered sets, which have rigid associate to them quiver.

Denote by $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $\mathcal{E} \in M_n(\mathbb{Z})$.

Definition 1. *An integer-valued matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called exponent matrix, if*

(1) $\alpha_{ii} = 0$ *for all* $i = 1, \ldots, n$;

(2) $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ *for* $1 \leq i, j, k \leq n$.

An exponent matrix $\mathcal{E}$ is said to be reduced, if $\alpha_{ij} + \alpha_{ji} > 0$, $i, j = 1, \ldots, n$; $i \neq j$.

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Put $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \ldots, n$, and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1 \leq k \leq n} (\beta_{ik} + \beta_{kj})$. Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is an $(0, 1)$-matrix. We call a quiver Q by simply laced if its adjacency matrix $[Q]$ is an (0,1)-matrix, i. e., in Q there is not multiple arrows and multiple loops.

The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of a strongly connected simply laced quiver $Q = Q(\mathcal{E})$.

Definition 2. *The quiver $Q(\mathcal{E})$ is called the quiver of a reduced exponent matrix $\mathcal{E}$.*

Definition 3. *An strongly connected simply laced quiver is said to be admissible, if it is the quiver of a reduced exponent matrix.*

Definition 4. *An admissible quiver Q is said to be rigid, if, up to equivalence, there exists a unique exponent matrix $\mathcal{E}$ such that $Q = Q(\mathcal{E})$.*

Let $Q = Q(X, U)$ be a quiver with the set of all vertices X and set of all arrows U. Denote by $b(u)$ the start vertex of arrow u and by $e(u)$ the end vertex of arrow u.

Theorem 1. *The admissible quiver $Q = Q(X, U)$, $|X| \geq 2$, in which all cycles pass through a single point, is rigid.*

Theorem 2. *If for the admissible quiver $Q = (X, U)$ there are two admissible quivers $Q_1 = (X_1, U_1)$ and $Q_2 = (X_2, U_2)$ such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, $U_i = \{(b(u), e(u)) | \{b(u), e(u)\} \subset X_i\}$, $i \in \{1, 2\}$, then the quiver $Q = (X, U)$ is not rigid.*

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Projective covers of modules over tiled order

V. N. Zhuravlyov

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{ij})\}$ be a tiled order, $M = \{\mathcal{O}, \mathcal{E}(M) = (\alpha_i)\}$ be a irreducible Λ-module, $M_1, \ldots, M_s$ be all maximal submodules of M and $\mathcal{E}(M_i) = \mathcal{E}(M) + e_{j_i}$, where $e_k = (\underbrace{0, \ldots, 0}, 1, 0, \ldots, 0)$. Denote by $P(M)$ the projective cover of M. Then $M = \sum\limits_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}$ and $P(M) = \bigoplus\limits_{i=1}^{s} \pi^{\alpha_{j_i}} P_{j_i}$.

Let $\varphi\colon P(M) \to M$ the epimorphism defined by formula $\varphi(m_1, \ldots, m_s) = m_1 + \ldots + m_s$, K be a kernel of epimorphism φ. For $l = 1, \ldots, s$ denote by K_l the kernel of epimorphism $\varphi_l\colon \bigoplus\limits_{i \neq l} \pi^{\alpha_{j_i}} P_{j_i} \to \sum\limits_{i \neq l} \pi^{\alpha_{j_i}} P_{j_i}$. Then the module $P(K)$ is a direct summand of $P(K_l) \oplus P\left(\sum\limits_{i \neq l} \pi^{\alpha_{j_i}} P_{j_i}\right)$.

Theorem. *If projective modules* $P\left(\sum\limits_{i \neq 1} \pi^{\alpha_{j_i}} P_{j_i}\right), \ldots, P\left(\sum\limits_{i \neq s} \pi^{\alpha_{j_i}} P_{j_i}\right)$ *do not have a common direct summand, then the module* $P(K)$ *does not have isomorphic direct summand.*

If projective modules $P\left(\sum\limits_{i \neq 1} \pi^{\alpha_{j_i}} P_{j_i}\right), \ldots, P\left(\sum\limits_{i \neq s} \pi^{\alpha_{j_i}} P_{j_i}\right)$ *have a common direct summand* P'*, then the module* $P(K)$ *have direct summand* $(P')^{n-1}$.

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Last Minute Abstracts

On the Piltz divisor problem over $\mathbb{Z}_{[i]}$ with convergence conditions

Popovych Polina

Let $k = p + q$, where p and q are natural numbers. We denote by $\tau_k^*(\alpha)$ the number of ways to represent the Gaussian integer α as a product of k factors, p of which satisfy certain convergence condition :

$$\tau_k^*(\alpha) = \#\{(\delta_1, \delta_2, \dots, \delta_k) \in \mathbb{Z}_{[i]} : \delta_1 \dots \delta_k = \alpha, \alpha_j \equiv \beta_j (mod \gamma_j), j = 1, \dots, p\},$$

where β_j, γ_j are given Gaussian integers with $N(\beta_j) < N(\gamma_j)$.

Let $F(s)$ be the generating function for $\tau_k^*(\alpha)$. Put

$$F_m(s) = N(\gamma)^s \sum_{\substack{\alpha \in \mathbb{Z}_{[i]} \\ \alpha \neq 0}} \frac{\tau_k^*(\alpha) e^{4miarg(\alpha)}}{N(\alpha)^s} = Z_m(s)^q \prod_{j=1}^{q} \xi_m(s, \delta_j),$$

where $Z_m(s)$ is the Hecke Zeta-function with Grossencharacter, $\lambda_m(\alpha) = e^{4miarg(\alpha)}$, $\xi_m(s, \delta_j) = \sum\limits_{\alpha} \frac{e^{4miarg(\alpha+\delta_j)}}{N(\alpha+\delta_j)^s}$, $Res > 1$, $m \in \mathbb{Z}$. Then we have

$$\sum_{N(\alpha) \leq x} \tau_k^*(\alpha) = \sum_{s=0,1} res \left(F_0(s) \left(\frac{x}{N(\gamma)} \right)^s \cdot \frac{1}{s} \right) + E_k(x),$$

where $\sum\limits_{s=0,1} res(F_0(s)(\frac{x}{N(\gamma)})^s \cdot \frac{1}{s})$ is the main term and $E_k(x)$ is the error term.

The object of the work is the construction of the upper and lower estimations of $E(x)$ and the analysis of the distribution of meanings for $\tau_k^*(\alpha)$ in sharp sectors: $S := \{\alpha : \varphi_1 < arg\alpha < \varphi_2\}$.

The paper is the generalization of the W. Nowak results.

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Diophantine equations with quadratic forms in thin sectors

Antonina Radova

In the works [1]-[2] the classic additive divisor problem is investigated

$$x_1x_2 - y_1y_2 = 1, \tag{1}$$

$x_1, x_2 < N, x_1, x_2 \in \mathbb{N}$.

In our paper we study an equivalent to the equation (1)

$$f(x,y,z) - \varphi(u,v) = 1, \tag{2}$$

$f(x,y,z) \leq N, x,y,z,u,v \in \mathbb{Z}$, where $f(x,y,z)$ (according $\varphi(u,v)$) is a positively definite quadratic form of three (according two) variables.

We shall build an asymptotic formula for the number $Q_s(N;k,l)$ of solutions of the equation (2), when

$$f(x,y,z) = x^2 + y^2 + z^2, \varphi(u,v) = u^2 + v^2, \tag{3}$$

$arg(u+iv) \in \mathbb{S}$, where $\mathbb{S}$ is a section $\acute{\alpha}_1 \leq \acute{\alpha}_2 \leq 1$, $\acute{\alpha}_1 - \acute{\alpha}_2 \to 0$ when $x \to \infty$ and values of quadratic form $\varphi(u,v) = u^2+v^2$ are l-free, $f(x,y,z) = x^2+y^2+z^2$ are k-free, $k \geq 2$, $l \geq 2$, M_t is the set of t-free numbers.

The number $Q_s(N;k,l)$ of solutions of the equation (3) is defined by the sum

$$Q_s(N;k,l) = \sum_{\substack{n=u^2+v^2 \leq N \\ arg(u+iv)\in\mathbb{S} \\ (n+1)\in M_k \\ n \in M_l}} r_3(n+1)r(n),$$

where $r(n)$ denotes the number of representations of the integer n by two squares and $r_3(n)$ denotes the number of representations of the integer n by three squares.

We prove the following

Theorem 1. *Let $Q_s(N;k,l)$ denote the number of the solutions of diophantine equations*

$$f(x,y,z) = \varphi(u,v) + 1,$$

$$f(x,y,z) \leq N, f(x,y,z) \in M_k, \varphi(u,v) \in M_l,$$

when $N \to \infty$ and $0 \leq \acute{\alpha}_1 - \acute{\alpha}_2 \leq N^{-\eta}$ the following asymptotic formula

$$Q_s(N;k,l) = \frac{2\alpha}{\pi} C_0 A(k,l) \prod_{p>2}(1 - \frac{p^2-1}{(p^2+1)p^k}) \prod_{p>2}(1 - \frac{g(k,l,p)}{p^l}) N^{3/2} + O(N^{5/4+1/4h+\varepsilon}),$$

where $C_0 > 0, A(k,l) > 0$ depends only of k,l, $\eta = \frac{1}{4} + \frac{1}{4h} + \varepsilon$, $C_0 = \frac{2\pi^2}{5}\frac{\xi(2)}{\xi(4)}$, $A(k,l) = (1 - \frac{3}{2^{k+1}})(1 - \frac{2^{k+2}}{2^l(5\cdot 2^k - 3)})$, $g(k,l,p) = \frac{p^k(p^2+1)(p+\chi_4(p))((p-\chi_4(p))g(l)+p\chi_4(p^l))}{p^l(p^{k+2}+p^k-p^2+1)(p^2+\chi_4^2(p))}$, $g(l) = \sum\limits_{u=0}^{l-1}\chi_4(p^u)$, $h = min(k,l) \geq 2$, $\varepsilon > 0$ holds.

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Index

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